DARBOUX TRANSFORMATIONS OF BISPECTRAL QUANTUM INTEGRABLE SYSTEMS

EMIL HOROZOV AND ALEX KASMAN

ABSTRACT. We present an approach to higher dimensional Darboux transformations suitable for application to quantum integrable systems and based on the bispectral property of partial differential operators. Specifically, working with the algebro-geometric definition of quantum integrability, we utilize the bispectral duality of quantum Hamiltonian systems to construct non-trivial Darboux transformations between completely integrable quantum systems. As an application, we are able to construct new quantum integrable systems as the Darboux transforms of trivial examples (such as symmetric products of one dimensional systems) or by Darboux transformation of well-known bispectral systems such as quantum Calogero-Moser.

1. INTRODUCTION

Given three ordinary differential operators L, K and $\tilde{L}$ we say that $\tilde{L}$ is the *Darboux transform* of L by K if they satisfy the intertwining relationship

$$(1) \qquad K \circ L = \tilde{L} \circ K$$

(after Darboux [8, 9] who considered such a transformation in the case that L is a one-dimensional Schrödinger operator). Since this transformation preserves certain classes of operators with distinguished spectral properties, it has been extremely useful in the investigation of integrable non-linear partial differential (*soliton*) equations (e.g., [3, 14, 17, 18]) and in the study of the bispectral property (e.g., [4, 10, 15]).

Due to the simplicity of the factorization properties of ordinary differential operators, it is easy to describe all possible Darboux transformations in the one dimensional case: *there exists an ordinary differential operator $\tilde{L}$ satisfying* (1) *if and only if $L \ker K \subset \ker K$.* For example, there is an intertwining relationship (1) for a particular ordinary differential operator L and a first order, monic dressing operator K

$$(2) \qquad K = f(x) \circ \partial \circ \frac{1}{f(x)}$$

if and only if $f(x)$ is an eigenfunction of the operator L.

Recently there has been much interest in the generalization of Darboux transformations to higher dimensional situations (e.g., [1, 5, 7, 12, 19]) where it has similar applications. In contrast to the one dimensional case where the ideas of the previous paragraph have universal application, results in the higher dimensional case are more limited. Consequently, papers on the subject can only describe certain classes of higher dimensional Darboux transformations and their applications. For example, [7] and [5] determine some relationships of the form (1) for partial differential operators, but only in the case where L is a constant coefficient operator;

[1, 12] consider the case in which L is a scalar Schrödinger operator intertwined by (1) with an operators having *matrix* coefficients (and the extension of this idea to more general base manifold); and [19] chooses an entirely different method to generalize Darboux transformations to higher dimensions, not involving (1). Most relevant to the present work is [5] where, in analogy to the form (2) from the one dimensional case, Darboux transformations are considered for constant coefficient partial differential operators by dressing operators of the form

$$(3) \quad K = p(x_1, \ldots, x_n) \circ q(\partial_1, \ldots, \partial_n) \circ \frac{1}{p(x_1, \ldots, x_n)} \qquad p, q \in \mathbb{C}[x_1, \ldots, x_n]$$

with q irreducible.

Although Darboux transformations have been frequently used to study the bispectral property, this paper conversely will demonstrate the use of bispectrality to construct Darboux transformations. An ordinary differential operator $L(x, \partial_x)$ in the variable x is said to be *bispectral* [10, 11] if it has a family of eigenfunctions $\psi(x, z)$

$$L\psi(x, z) = f(z)\psi(x, z)$$

parametrized by the spectral parameter z (meaning that $f(z)$ is a non-constant function) which is also an eigenfunction for an ordinary differential operator $\Lambda(z, \partial_z)$ in z with x playing the role of spectral parameter

$$\Lambda\psi(x, z) = \phi(x)\psi(x, z).$$

A useful observation (cf. [2]) is that the algebra B generated by L and $\phi(x)$ and the algebra B' generated by Λ and $f(z)$ are related by an *anti-isomorphism*

$$
\begin{aligned}
(4) \qquad\qquad b : B \quad &\rightarrow \quad B' \\
(5) \qquad\qquad L \quad &\mapsto \quad f(z) \\
(6) \qquad\qquad \phi(x) \quad &\mapsto \quad \Lambda \\
(7) \qquad\qquad b(c + d) \quad &= \quad b(c) + b(d) \\
(8) \qquad\qquad b(cd) \quad &= \quad b(d)b(c) \\
(9) \qquad\qquad b(c) \quad &= \quad 0 \text{ iff } c = 0.
\end{aligned}
$$

After defining the notions of Darboux transformations and bispectrality for quantum Hamiltonian systems in the algebro-geometric context [6], we demonstrate that the corresponding anti-isomorphism b (cf. (4)-(9)) allows the construction of intertwining relationships between (non-constant coefficient) commutative rings of partial differential operators (Theorem 5.1). As an application, new, non-trivial examples of quantum integrable systems can be constructed by Darboux transformation of trivial examples and by Darboux transformation of known bispectral examples such as the quantum Calogero-Moser system.

Although it is not necessary to the understanding of the present paper, we would like to point out that the definition of Darboux transformation for quantum integrable systems presented below is inspired by the constructions in a number of papers [3, 4, 5, 13, 14, 16] which generalize and interpret the results of G. Wilson [21] in terms of Darboux transformations of rings of ordinary differential operators

2. QUANTUM INTEGRABLE SYSTEMS

As usual, an integrable system is defined by the presence of "sufficiently many" commuting Hamiltonians. In the algebro-geometric context, when the configuration

space happens to be an algebraic variety, there is a convenient geometric description of classical integrable systems. In this situation such a system is given by an embedding of the coordinate ring of another variety Y of the *same dimension* as a Poisson commutative subring of the coordinate ring of the cotangent bundle to X.

As explained in [6], the quantum analogue of this construction is to view the Hamiltonian system as a commutative subring of the ring of differential operators on the configuration variety X. In particular, we define a Quantum Hamiltonian System (QHS) on a smooth, irreducible, affine algebraic variety X (over the algebraically closed field $\mathbb{F}$ in characteristic zero) to be a pair $S = (\Lambda, L)$ where Λ is also an affine variety over $\mathbb{F}$ and

$$(10) \qquad L : \mathcal{O}(\Lambda) \quad \to \quad \mathcal{D}(X)$$
$$(11) \qquad\qquad\qquad f \quad \mapsto \quad L_f$$

is an embedding of the coordinate ring of Λ into the ring of differential operators on X. The system S is said to be a Completely Integrable Quantum System (CIQS) if $\dim \Lambda = \dim X$. These systems are of particular interest since they can be completely analyzed.

Associated to S is a family of $\mathcal{D}$-modules parametrized by point $\lambda \in \Lambda$ having generators 1_λ satisfying

$$L_g \cdot 1_\lambda = g(\lambda)1_\lambda \qquad \forall g \in \mathcal{O}(\Lambda).$$

One may think of this $\mathcal{D}$-module as a system of differential equations

$$L_g \cdot \psi = g(\lambda)\psi \qquad \forall g \in \mathcal{O}(\Lambda)$$

which is known as the spectral problem associated to S [6].

Example 2a: The simplest example is S_W^n, which is the completely integrable system with $\Lambda = X = \mathbb{C}^n$ given by $L_p = p(\partial_1, \dots, \partial_n)$ for $p \in \mathbb{C}[x_1, \dots, x_n] = \mathcal{O}(X)$ and $\partial_i = \frac{\partial}{\partial x_i}$. Note that S_W^n is actually nothing but the symmetric product of S_W^1 with itself n times.

Example 2b: Let $\mathcal{A}$ be a finite set of vectors $\alpha \in \mathbb{C}^k$ and $m_\alpha \in \mathbb{N}$ be a set of positive integers for $\alpha \in \mathcal{A}$. Let $L = -\Delta + u(x)$ $(x = (x_1, \dots, x_k))$ be the Schrödinger operator where $\Delta = \partial_1^2 + \partial_2^2 + \cdots + \partial_k^2$ with $\partial_j = \frac{\partial}{\partial x_j}$ and

$$u(x) = \sum_{\alpha \in \mathcal{A}} \frac{m_\alpha(m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2}.$$

In [20] and the literature cited therein one can find necessary and sufficient conditions on $\mathcal{A}$ and m_α such that a wave function $\Psi(x, z)$ exists $(z = (z_1, \dots, z_n))$ with the property that $L\Psi = -||z||^2\Psi$. Let f be any polynomial in $\mathbb{C}[x_1, \dots, x_k]$ satisfying

$$\partial_\alpha f = \partial_\alpha^3 f = \cdots = \partial_\alpha^{2m_\alpha - 1} f = 0$$

for any x on the hyperplane $(\alpha, x) = 0$ for each $\alpha \in \mathcal{A}$. Then there corresponds a differential operator L_f in x such that $L_f \Psi = f(z)\Psi$. Moreover, the coefficients of L_f are rational in x having singularities only on the vanishing set of the polynomal $p = \prod(x, \alpha)$. So, in particular, one may choose any sufficiently large finitely generated subring A of such polynomials f and consider the completely integrable quantum system $S = (\text{Spec } A, L)$ on the quasi-affine variety X which is the localization of $\mathbb{C}^k$ by p.

2.1. Localization. Given a QHS $S = (\Lambda, L)$ on the variety X and an embedding $\gamma : \mathcal{D}(X) \to \mathcal{D}(Y)$ of the ring of differential operators on X into the ring of operators on some *other* variety Y, one obviously can define a "new" quantum Hamiltonian system $\hat{S} = (\Lambda, \gamma \circ L)$ on Y. In general, one would not be interested in doing so, since $\hat{S}$ is equivalent to S in almost every way. However, a particular example of this construction will be used in the following, and so it will be useful to develop an appropriate notation here.

Let X be an irreducible affine variety and $f \in \mathcal{O}(X)$ a non-zero function in the coordinate ring. The localization $\mathcal{O}_f$ constructed from $\mathcal{O}(X)$ by formally introducing the inverse $\frac{1}{f}$ is the coordinate ring of the open subvariety $X_f = \mathrm{Spec}\,(\mathcal{O}_f)$. In particular, $\dim X_f = \dim X$. One may extend the action of $\mathcal{D}(X)$ to $\mathcal{O}(X_f)$ using the Leibniz rule and thus view $\mathcal{D}(X)$ as a subring of $\mathcal{D}(X_f)$ embedded merely by the inclusion γ_f.

Definition: Given $S = (\Lambda, L)$ on X and $f \neq 0 \in \mathcal{O}(X)$ define the localization of S by f to be the QHS $S_f = (\Lambda, \gamma_f \circ L)$ QHS on X_f.

In order to simplify notation, we will make use of the embedding γ_f to identify $\mathcal{D}(X)$ with its image in $\mathcal{D}(X_f)$. That is, we will write $c \in \mathcal{D}(X_f)$ for any $c \in \mathcal{D}(X)$ rather than writing $\gamma_f(c)$.

3. Darboux Transformations for QHSs

Let $S = (\Lambda, L)$ be a QHS on X. For any choice of $K \in \mathcal{D}(X)$ let $A_K \subset \mathcal{O}(\Lambda)$ be defined as

$$(12) \qquad A_K = \{f \in \mathcal{O}(\Lambda) | K \circ L_f = \tilde{L}_f \circ K \text{ for some } \tilde{L}_f \in \mathcal{D}(X)\}.$$

In general, one would expect that $A_K = \mathbb{F}$ (i.e., A_K contains only constant functions). However, if this is not the case, then K may be used as a "Darboux dressing" to construct a new quantum system as follows.

Let $A \subset A_K$ be a finitely generated subring. Then define the *Darboux transform* $\tilde{S} = DT(S, K, A)$ to be the pair $\tilde{S} = (\tilde{\Lambda}, \tilde{L})$ where $\tilde{\Lambda} = \mathrm{Spec}\,A$ and $\tilde{L}_f$ is defined by equation (12). It is then elementary to verify that $\tilde{S}$ is also a QHS on X.

Clearly, the difficulty then is in choosing an appropriate operator K. Although later in this paper we *will* describe below a procedure for choosing a dressing operator K leading to a non-trivial Darboux transformation, thus far we have not given any way in which to determine non-trivial Darboux transformations of QHSs in dimension greater than one. Below we will recall two examples from earlier works which demonstrate Darboux transformations between integrable systems.

Example 3a: It is shown in [7] that certain quantum Calogero models are Darboux transformations (according to the definition above) of a localization of the quantum system S_W^n (cf. Example 2a). We will briefly recall here the explicit computation in Section 3 of [7] and "translate" it into the notation used above. Let S_p be the localization of $S = S_W^3$ by the polynomial[1] $p(x_1, x_2, x_3) = x_{12}x_{13}x_{23}$. Then, one may consider the Darboux transformation $\tilde{S}_p = DT(S_p, K, A)$ where

$$
\begin{aligned}
K \;=\; & \partial_{12}\partial_{13}\partial_{23} - 2x_{12}^{-1}\partial_{13}\partial_{23} - 2x_{13}^{-1}\partial_{12}\partial_{23} - 2x_{23}^{-1}\partial_{12}\partial_{13} \\
& + 4x_{23}^{-1}x_{13}^{-1}\partial_{12} + 4x_{13}^{-1}x_{12}^{-1}\partial_{23} + 4x_{12}^{-1}x_{23}^{-1}\partial_{13} - 12x_{12}^{-1}x_{13}^{-1}x_{23}^{-1}
\end{aligned}
$$

---

[1] Here we use the notation $x_{ij} = x_i - x_j$ and $\partial_{ij} = \partial_i - \partial_j$.

and where $A \subset \mathbb{C}[x_1, x_2, x_3]$ is the ring $\mathbb{C}[h_1, h_2, h_3]$ with $h_i = x_1^i + x_2^i + x_3^i$. Then $\tilde{S}$ is the three particle quantum rational Calogero-Moser system. In fact, $\tilde{L}_{h_2} = \Delta - 4 \sum_{i<j} x_{ij}^{-2}$ is the standard Hamiltonian of this well known integrable system. (This is, in fact, a special case of Example 2b.)

Example 3b: The first example in Section 5 of [5] is a Darboux transformation of a localization of $S = S_W^2$. Let

$$K = \left(\partial_1 - \frac{2x_1}{x_1^2 - x_2} \right) \left(\partial_1 + \frac{1}{x_1^2 - x_2} \right) - \lambda$$

$$\tau(x_1, x_2) = x_1^2 - x_2 \qquad q(x_1, x_2) = x_1 x_2 - \lambda \in \mathbb{C}[x_1, x_2] = \mathcal{O}(\Lambda)$$

and $A = \mathbb{C}[q^3, x_1 q^3, x_2 q^3]$. Then, it follows from the results of [5] that for any fixed $\lambda \in \mathbb{C}$, the Darboux transform $DT(S_\tau, K, A)$ is also a quantum integrable system.

4. BISPECTRALLY DUAL QHSs

Consider a pair of quantum Hamiltonian systems (all varieties defined over $\mathbb{F}$):

$$S = (\Lambda, L) \qquad \text{on } X$$

$$S' = (\Lambda', L') \qquad \text{on } X'.$$

Suppose that X' covers Λ and that X covers Λ', giving us natural embeddings

$$\pi : \mathcal{O}(\Lambda) \to \mathcal{O}(X') \qquad \pi' : \mathcal{O}(\Lambda') \to \mathcal{O}(X).$$

Given $f \in \mathcal{O}(\Lambda)$ and $g \in \mathcal{O}(\Lambda')$ we have both $L_f \in \mathcal{D}(X)$ and $\pi'(g) \in \mathcal{D}(X)$. So, in particular, by using L and π' we can view $\mathcal{O}(\Lambda)$ and $\mathcal{O}(\Lambda')$ as being subrings of $\mathcal{D}(X)$. Similarly, L' and π allow us to consider both coordinate rings as subrings of $D(X')$.

Definition: Let $B \subset \mathcal{D}(X)$ be the subring generated by the images of L and π'. Similarly, let $B' \subset D(X')$ be the subring generated by the images of L' and π. We say that S and S' are *bispectrally dual* to each other if there exists an anti-isomorphism $b : B \to B'$ such that

1. $b(L_f) = \pi(f) \qquad \forall f \in \mathcal{O}(\Lambda)$
2. $b(\pi'(g)) = L'_g \qquad \forall g \in \mathcal{O}(\Lambda')$
3. $b(L_1 L_2) = b(L_2) b(L_1)$ for all $L_1, L_2 \in B$
4. $b(L_1 + L_2) = b(L_1) + b(L_2)$ for all $L_1, L_2 \in B$
5. $b(L) = 0$ iff $L = 0 \in B$.

Note: In order to simplify notation, we will use the embeddings π and π' to consider their images as actually being contained in the coordinate rings of the covering varieties. In particular, we will write $g \in \mathcal{O}(X) \subset \mathcal{D}(X)$ for any $g \in \mathcal{O}(\Lambda')$ rather than $\pi'(g)$ and similarly $f \in \mathcal{O}(X') \subset \mathcal{D}(X')$ for any $f \in \mathcal{O}(\Lambda)$.

The following lemma, whose statement is well known at least in the case of bispectral ordinary differential operators [10, 21] demonstrates that the existence of the anti-isomorphism is a severe restriction.

Lemma 4.1. *Denote by A_i and A_i' (for $i \in \mathbb{N}$) the differential operators*

$$A_i = \mathrm{ad}_{L_f}^i (g) \in \mathcal{D}(X) \qquad A_i' = (-1)^i \mathrm{ad}_f^i \left(L_g' \right) \in \mathcal{D}(X').$$

Then it follows from the bispectral duality that $b(A_i) = A_i'$ for all $i \in \mathbb{N}$ and, in particular, that $A_i = 0$ for $i > \mathrm{ord}\, L_g'$.

Proof. When $i = 0$ we simply have the known property $b(\pi'(g)) = L'_g$ of the map b. Then, supposing that the claim is known to apply for i we check that

$$
\begin{aligned}
b(A_{i+1}) &= b(L_f \circ A_i - A_i \circ L_f) \\
&= A'_i \circ f - f \circ A'_i \\
&= -\mathrm{ad}_f A'_i = A'_{i+1}.
\end{aligned}
$$

Since commutation with a function lowers the order of an operator, it is clear that $A'_i = 0$ for any $i > \mathrm{ord}\, L'_g$. However, the anti-isomorphism demonstrates that $A_i = 0$ for $i > \mathrm{ord}\, L'_g$ as well. $\qquad\square$

In general, for a differential operator L of order greater than 0 and differential operator g of order 0, one would expect that $\mathrm{ad}^i_L g$ would be an operator of high order for arbitrarily large $i \in \mathbb{N}$. The fact that in the bispectral case the order is 0 demonstrates that bispectral duality is an unusual situation. However, many well known examples of quantum integrable systems are related by this duality.

As described in [2], whenever one has a bispectral situation – commutative rings of operators in the variables x and z respectively sharing a common eigenfunction $\psi(x, z)$ with eigenvalues depending on z and x respectively – there automatically exists an anti-isomorphism b between the algebras generated by the operators and eigenvalues in the separate variables. Specifically, b is the map defined by the equation $L\psi(x, z) = b(L)\psi(x, z)$ for all $L \in B$. In each of the examples below, the existence of such a common eigenfunction is used to demonstrate the bispectral duality of certain quantum integrable systems.

Example 4a: The differential operators from the system S^n_W are precisely the ring of constant coefficient differential operators in the variables $x = (x_1, \ldots, x_n)$. In particular, they have the common eigenfunction $\Psi(x, z) = e^{(x,z)}$ ($z = (z_1, \ldots, z_n)$). Then, since this eigenfunction is shared also by the constant coefficient operators in z, we may conclude that S^n_W is bispectrally dual to *itself*. In particular, letting both S and S' be copies of S^n_W one finds that B is the Weyl algebra and b is the usual anti-isomorphism.

Example 4b: Similarly, since the eigenfunction $\Psi(x, z)$ from Example 2b is symmetric in x and z [20], again one finds that these systems are all bispectrally self-dual.

Example 4c: As described in [5], the CIQS $\tilde{S}_\tau$ on the local variety $\mathbb{C}^2_\tau$ from Example 3b is not-self dual, but is bispectrally dual to a system $\tilde{S}'_\tau = (\tilde{\Lambda}', \tilde{L}')$ on $\mathbb{C}^2_{x_1 x_2 - \lambda}$.

Example 4d: Since any commutative ring of bispectral differential operators may be viewed as an example of a bispectral quantum integrable system on a quasi-affine variety, one may find many additional examples in the papers [2, 4, 5, 13, 21].

Remark: Given completely integrable quantum systems S_i bispectrally dual to the systems S'_i ($1 \le i \le n$) the *symmetric products* are bispectrally dual. So, in particular, one may take *any* set of bispectral ordinary differential operators (viewed as operators in different variables) and this gives (trivial) bispectral quantum integrable systems in any number of variables. This is significant because the techniques of the next section will allow us to transform these trivial examples into non-trivial examples.

5. Bispectral Darboux Transformations

Let $S = (\Lambda, L)$ on X and $S' = (\Lambda', L')$ on X' be *bispectrally dual completely integrable* quantum systems. Let us fix an arbitrary choice of $f \in \mathcal{O}(\Lambda)$ and $g \in \mathcal{O}(\Lambda')$. Associated to this choice of a pair of functions we will construct by Darboux transformation a new bispectrally dual pair of completely integrable quantum systems: $\tilde{S} = (\tilde{\Lambda}, \tilde{L})$ on the localized variety[2] X_g and $\tilde{S}' = (\tilde{\Lambda}', \tilde{L}')$ on X'_f. Since the two Darboux dressing operators to be used are related by the anti-isomorphism b, this is an example of a *bispectral Darboux transformation* [2, 13, 15].

The key observation which will allow the Darboux transformations to be performed are the following factorizations which can be achieved in the subring $B \subset \mathcal{D}(X)$:

Lemma 5.1. *Let f and g be as above, let $m = \operatorname{ord} L_f$ and $n = \operatorname{ord} L'_g$, then there exist elements $K, Q, R \in B$ satisfying*

$$(13) \qquad\qquad g^{m+1} \circ L_f = K \circ g$$

$$(14) \qquad\qquad L_f \circ g^{m+1} = g \circ R$$

$$(15) \qquad\qquad L_f^{n+1} \circ g = Q \circ L_f$$

Proof. Denote by I the left ideal $I = Bg \subset B$. One has

$$g \circ L_f = L_f \circ g + [g, L_f]$$

i.e.

$$g \circ L_f = \operatorname{ad}_g L_f \bmod I$$

where $\operatorname{ad}_g L_f = [g, L_f] \in B$ and has order at most $m - 1$.

By the same argument, we can see that $g^2 \circ L_f = \operatorname{ad}_g^2 L_f \bmod I$ where again $\operatorname{ad}_g^2 L_f \in B$ and is an operator of order $m - 2$ or less. Continuing the procedure one gets that

$$g^{m+1} \circ L_f = \operatorname{ad}_g^{m+1} L \bmod I = 0 \bmod I.$$

Consequently we conclude that $g^{m+1} \circ L_f \in I$ which proves the claim.

Equation (14) is proved in the same manner using instead the right ideal gB.

Moreover, by symmetry of definition we get equivalent factorization in B'. In particular, we see that $L'_g \circ f^{n+1} = f \circ R'$ for some $R' \in B'$. Letting $Q = b(R')$ and applying the anti-isomorphism b to this equation yields (15). $\qquad\square$

The main result then is that one may use the operator K from Lemma 5.1 as a Darboux dressing operator for the localized system S_g and one may use $b(K)$ as a Darboux dressing operator for the localized system S'_f and that the resulting systems are also bispectrally dual quantum integrable systems.

Theorem 5.1. (a) *Let $\{e_1, \dots, e_N\}$ be a complete set of generators of $\mathcal{O}(\Lambda)$ over $\mathbb{F}$ and denote by A the subring generated by $\{f^{n+1}, f^{n+1}e_1, \dots, f^{n+1}e_N\}$ where $n = \operatorname{ord} L'_g$. Then if K is the operator defined in (13) the Darboux transform*

$$\tilde{S} = (\tilde{\Lambda}, \tilde{L}) = DT(S_g, K, A)$$

is a completely integrable quantum system on X_g.

---

[2]Recall that we are utilizing the embedding π' to consider $g(= \pi'(g))$ as an element of $\mathcal{O}(X)$.

(b) *Let $\{e'_1, \ldots, e'_M\}$ be a complete set of generators of $\mathcal{O}(\Lambda')$ and denote by A' the subring generated by $\{g^{m+1}, e'_1 g^{m+1}, \ldots, g^{m+1} e'_M\}$ where $m = \operatorname{ord} L_f$. Then the Darboux transform*

$$\tilde{S}' = (\tilde{\Lambda}', \tilde{L}') = DT(S'_f, b(K), A')$$

is a completely integrable quantum system on X'_f.

(c) *The systems $\tilde{S}$ and $\tilde{S}'$ are bispectrally dual.*

Proof. (a) We must first show that A is a subring of A_K (cf. (12)) so that the Darboux transformation is well defined. Note that by (15) we have that

$$L_f^{n+1} = Q \circ L_f \circ g^{-1} \in \mathcal{D}(X_g).$$

Moreover, from (13) we see that

$$L_f \circ g^{-1} = g^{-m-1} \circ K \in \mathcal{D}(X_g).$$

Combining these we conclude that

$$L_f^{n+1} = Q \circ g^{-m-1} \circ K.$$

Since every element of A has a factor of f^{n+1} (by definition), for any $h \in A$ we have that $L_h = L_{h'} \circ L_f^{n+1}$ for some $h' \in \mathcal{O}(\Lambda)$. Consequently one has

$$K \circ L_h = \tilde{L}_h \circ K$$

with

$$(16) \qquad \tilde{L}_h = K \circ L_{h'} \circ Q \circ g^{-m-1}.$$

Thus, it follows that $\tilde{S} = (\tilde{\Lambda}, \tilde{L}) = DT(S_g, K, A)$ is a well defined QHS with $\tilde{\Lambda} = \operatorname{Spec} A$ and $\tilde{L}$ defined by (16).

To see furthermore that this is an *integrable* system, it suffices to note that the quotient field of the varieties Λ and $\tilde{\Lambda}$ are isomorphic. Consequently, the varieties are birational and their dimensions are equal. (Essentially, we have introduced singularities along $f^{-1}(0) \subset \Lambda$.)

(b) The proof of the second statement is the same, based on the observation that applying the anti-isomorphism b to (13) gives

$$L_g'^{m+1} = f^{-1} \circ L'_g \circ b(K) \in \mathcal{D}(X'_f).$$

(c) We wish to demonstrate the existence of an anti-isomorphism $\tilde{b}$ between the ring $\tilde{B}$ generated by the operators $\tilde{L}_v$ and functions w for $v \in \mathcal{O}(\tilde{\Lambda})$ and $w \in \mathcal{O}(\tilde{\Lambda}')$ and the ring $\tilde{B}'$ generated by the operators $\tilde{L}'_w$ and functions v. This follows automatically from the existence of b since we have conjugated by K and $b(K)$ respectively. In particular, one may formally (cf. [2]) define $\tilde{b}$ by its action on the generators: $\tilde{b}(\tilde{L}_v) = b(K^{-1} \circ \tilde{L}_v \circ K) = b(L_v) = v$ and $\tilde{b}(w) = b(K^{-1}wK) = b(K)L'_w b(K)^{-1}$. Then the fact that it extends to an anti-isomorphism on all of $\tilde{B}$ and $\tilde{B}'$ follows from the fact that it is just a conjugation of the anti-isomorphism b. $\square$

Remark: Since this procedure takes any bispectral quantum integrable systems to a new pair of the same type, it may be iterated indefinitely to produce new examples.

Example 5a: Let $\Lambda = X = \mathbb{C}^2$ and consider the CIQS $S = (\Lambda, L)$ given by $L_p = p(\partial_1, \partial_2^2 - x_2)$ for $p \in \mathbb{C}[x_1, x_2]$. The images of the generators x_1 and x_2 trivially commute since they are simply ordinary differential operators in separate variables. In this way, we see that this is a trivial example since it is merely the symmetric product of two one dimensional examples. Moreover, since the common eigenfunction $\psi(x, z) = e^{x_1 z_1} Ai(x_2 + z_2)$ satisfying $L_p \psi = p\psi$ is symmetric in x and z, this example is also trivially self-dual. (So, $b(p) = L_p$ and $b(L_p) = p$ defines the necessary anti-isomorphism of the Weyl algebra in two variables.) However, we may use Darboux transformation to produce a new quantum system from this one whose integrability and duality are not obvious.

Consider $f = x_1^2 + x_2$ and $g = x_1 + s x_2 \in \mathbb{C}[x_1, x_2] = \mathcal{O}(\Lambda)$, then $\operatorname{ord} L_f = \operatorname{ord} L_g = 2$. The results above allow us to consider the Darboux transformation $\tilde{S}$ of the localization S_g by the dressing operator

$$
\begin{aligned}
K &= g^{m+1} \circ L_f \circ g^{-1} \\
&= (x_1 + s x_2)^2 (\partial_1^2 + \partial_2^2) - 2(x_1 + s x_2)(\partial_1 + s \partial_2) + [(x_1 + s x_2)^2 + 2 + 2s^2].
\end{aligned}
$$

Specifically, in terms of the generators $e_1 = 1$, $e_2 = x_1$, $e_3 = x_2$ one finds that $\tilde{S} = (\tilde{\Lambda}, \tilde{L})$ where $\tilde{\Lambda} = \operatorname{Spec} \mathbb{C}[\tilde{e}_1, \tilde{e}_2, \tilde{e}_3]$ ($\tilde{e}_i := f^2 e_i$) is a two dimensional singular rational variety and $\tilde{L}$ is defined by its action on the generators

$$
\tilde{L}_{\tilde{e}_i} = K \circ L_{e_i} \circ Q \circ g^{-3}
$$

with

$$
Q = L_f^2 \circ (x_1 + s x_2) + 2 L_f \circ (\partial_1 + s \partial_2) + 2.
$$

Moreover, one may also transform the system S' localized at f using the dressing operator $b(K)$ to get the system $\tilde{S}' = (\tilde{\Lambda}', \tilde{L}')$ where $\tilde{\Lambda}' = \operatorname{Spec} \mathbb{C}[\tilde{e}_1', \tilde{e}_2', \tilde{e}_3']$ ($\tilde{e}_i' := g^3 e_i$) is also a two dimensional singular rational variety and $\tilde{L}'$ is defined by its action on the generators

$$
\tilde{L}'_{\tilde{e}_i'} = b(K) \circ L_{e_i} \circ f^{-1} \circ L_g.
$$

The fact that $\tilde{S}$ and $\tilde{S}'$ are bispectrally dual is demonstrated by the fact that the eigenfunctions $\tilde{\psi}(x, z) := K\psi(x, z)$ and $\tilde{\psi}'(x, z) := b(K)\psi(x, z)$ satisfying

$$
\tilde{L}_p \tilde{\psi} = p(z)\tilde{\psi} \qquad \tilde{L}'_q \tilde{\psi}' = q(z)\tilde{\psi}' \qquad \forall p \in \mathcal{O}(\Lambda) \; q \in \mathcal{O}(\Lambda')
$$

are related by the exchange of x and z:

$$
\tilde{\psi}(x, z) = \tilde{\psi}'(z, x).
$$

Example 5b: Of course, one may also use the construction described above to determine completely integrable Darboux transforms of the quantum Calogero-Moser system given in Example 3a. For example, it now follows that one may use $K = h_i^3 \circ \tilde{L}_{h_2} \circ h_i$ as a Darboux dressing operator for this system to determine a new quantum integrable system (having an operator of order $2i + 2$ as its lowest order Hamiltonian).

References

[1] A. Andrianov, N. Borisov and M. Ioffe, "The factorization method and quantum systems with equivalent energy spectra", Physics Letters A, 105 (1984) no 1,2 pp. 19–22.

[2] B. Bakalov, E. Horozov, and M. Yakimov, "General Methods for Constructing Bispectral Operators" *Physics Letters A* 222 (1996) pp. 59–66

[3] B. Bakalov, E. Horozov, M. Yakimov, "Bäcklund-Darboux transformations in Sato's grassmannian, Serdica Math. Jour. 22 (1996) 571-588.

[4] B. Bakalov, E. Horozov, M. Yakimov, "Bispectral commutative rings of ordinary differential operators", *Comm. Math. Phys.* 190 (1997) pp. 331–373

[5] Yu. Berest and A. Kasman, "$\mathcal{D}$-modules and Darboux transformations", *Letters in Mathematical Physics* 43 (1998) pp. 279–294

[6] A. Braverman, P. Etingof and D. Gaitsgory, "Quantum Integrable Systems and Differential Galois Theory", *Transformation Groups* 2 (1997) pp. 31–56

[7] O.A. Chalykh and A.P. Veselov, "Commutative rings of partial differential operators and Lie algebras", Comm. Math. Phys. 125 (1990) 597-611.

[8] G. Darboux , "Sur une proposition relative aux equation lineaires" Compt. Rend. **94** (1882), 1456

[9] G. Darboux, "Leçons sur la théorie générale des surfaces." 2éme partie, Paris, Gauthiers-Villars, 1889.

[10] J.J. Duistermaat and F.A. Grünbaum, "Differential equations in the spectral parameter", Comm. Math. Phys. 103 (1986) 177-240

[11] F.A. Grünbaum, "Bispectral Musings" CRM Proceedings and Lecture Notes, Volume 14, American Mathematical Society, Providence, RI, (1998) pp. 31–46.

[12] A. Gonzalez-Lopez and N. Kamran, "The Multidimensional Darboux Transformation", hep-th/961200.

[13] A. Kasman and M. Rothstein, "Bispectral darboux transformations: the generalized Airy case", Physica D 102 (1997) p. 159-173.

[14] A. Kasman, "Darboux transformations from n-KdV to KP", *Acta Applicandae Mathematicae* 49 no. 2 (1997) pp. 179–197.

[15] A. Kasman, "Darboux transformations and the bispectral problem", *CRM Proceedings and Lecture Notes*, 14, American Mathematical Society, Providence, RI (1998) pp. 81–91.

[16] J. Liberati, "Bispectral property, Darboux transformation and the Grassmannian Gr$^{\text{rat}}$" *Lett. Math. Phys.* 41 (1997), no. 4, pp. 321–332.

[17] V.B. Matveev, "Darboux transformation and explicit solutions of the Kadomtcev-Petviaschvily Equation, Depending on Functional Parameters" *Letters in Mathematical Physics* 3 (1979), pp. 213-216

[18] V.B. Matveev and M.A. Salle, *Darboux Transformations and Solitons*, Springer-Verlag, Berlin, 1991.

[19] P. Sabatier, "On multidimensional Darboux transformations", Inverse Problems, 14 no. 2 (1998) pp. 355-366

[20] A.P. Veselov, "Baker-Akhiezer Functions and the Bispectral Problem in Many Dimensions", *CRM Proceedings and Lecture Notes* American Mathematical Society, Providence, RI, 1998 pp. 123–127

[21] G. Wilson, "Bispectral commutative ordinary differential operators", J. Reine angew. Math. 442 (1993) 177-204

Department of Mathematics and Informatics, University of Sofia, Bulgaria

Department of Mathematics, College of Charleston, Charleston, SC