DARBOUX TRANSFORMATIONS FROM n-KDV TO KP

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Abstract. The iterated Darboux transformations of an ordinary differential operator are constructively parametrized by an infinite dimensional grassmannian of finitely supported distributions. In the case that the operator depends on time parameters so that it is a solution to the n-KdV hierarchy, it is shown that the transformation produces a solution of the KP hierarchy. The standard definitions of the theory of τ-functions are applied to this grassmannian and it is shown that these new τ-functions are quotients of KP τ-functions. The application of this procedure for the construction of "higher rank" KP solutions is discussed.

1. Introduction

The Darboux transformation is a technique for producing a new differential operator and eigenfunction from a known operator eigenfunction pair. Essentially, the technique involves factoring the operator and exchanging its factors. Equivalently, the transformation can be seen as a conjugation of the operator by another differential operator. Originally used by Darboux in the context of the stationary Schrödinger equation, this technique has become an important tool in the study of integrable non-linear evolution, or soliton, equations.

Another technique frequently used in the study of soliton equations is the association, through the work of Burchnall and Chaundy [4], of algebro-geometric objects including an algebraic (spectral) curve and a vector bundle to commutative rings of ordinary differential operators [11]. In this construction, the common eigenfunctions of the elements of the ring are viewed as a vector bundle over the algebraic curve of common eignvalues of the ring. The rank of the bundle is the greatest common divisor of the orders of the elements of the ring, and is referred to as the rank of the ring. Then, the ring behaves asymptotically near the point at infinity like the polynomial ring $\mathbb{C}[L_0]$ for some differential operator L_0 whose order is the rank of the ring [20].

The present paper investigates an infinite dimensional grassmannian $Gr(\mathbb{D}^n)$ whose points determine iterated Darboux transformations of the ring $\mathbb{C}[L_0]$ into a commutative ring of ordinary differential operators with vacuum L_0 for arbitrary choices of L_0 with order n. Specifically, given an ordinary differential operator L_0 of order n, the choice of a point $C \in Gr(\mathbb{D}^n)$ determines a subring $A_C \subset \mathbb{C}[z]$ and an ordinary differential operator K_C such that

$$K_C p(L_0) = Q_p K_C$$

for any $p \in A_C$ and some ordinary differential operator Q_p. Thus, Q_p is achieved as a Darboux transformation of $p(L_0)$ and

$$\mathbf{R}_C = \{Q_p | p \in A_C\}$$

is a commutative ring with rank n and vacuum operator L_0.

In the case that the vacuum operator L_0 is chosen to satisfy the equations of the n-KdV hierarchy

$$\frac{\partial}{\partial t_i} L_0 = [(L_0^{i/n})_+, L_0]$$

then pseudo-differential operator

$$\mathbf{L}_C = K_C L_0^{1/n} K_C^{-1}$$

is seen to be a solution of the KP hierarchy. In this way, the results described above can be seen as an investigation of Darboux transformations taking solutions of n-KdV to solutions of KP and therefore as a generalization of the results of Latham-Previato [16] on Darboux transformations from solutions of the KdV *equation* to the KP equation. The methods used are largely based on those developed in [25] for the case $n = 1$.

2. Distributions

Let $\mathbb{D}$ denote the set of finitely supported distributions. That is, elements of $\mathbb{D}$ are finite linear combinations of functionals of the form

$$\delta_\lambda \circ \partial_z^i \qquad \lambda \in \mathbb{C},\ i \in \mathbb{N}$$

acting on functions in the variable z (sufficiently differentiable at $z = \lambda$) by

$$\delta_\lambda \circ \partial_z^i \left(f(z)\right) = f^{(i)}(\lambda).$$

Then, elements of $\mathbb{D}^n$ can be interpreted as linear functionals on z dependent n-vectors:

$$c(f_1, \ldots, f_n) = \sum_{i=1}^n c_i(f_i) \qquad c \in \mathbb{D}^n.$$

Subspaces of $\mathbb{D}$ were used to construct explicit soliton solutions to the KP hierarchy in [22] and then to construct rational solutions in

[25]. The theory to be developed below is a generalization of these techniques using subspaces of $\mathbb{D}^n$.

For any subspace $\mathbf{V} \subset \mathbb{D}^n$ and natural number $N < \dim \mathbf{V}$ one may consider the grassmannian

$$Gr_N(\mathbf{V}) := \{C \subset \mathbf{V} | \dim W = N\} \tag{1}$$

of N dimensional subspaces of $\mathbf{V}$. In the present paper, we will only be interested in the case $N \equiv 0 \mod n$ and so this relationship will be assumed in all that follows. It will further be convenient to consider the set

$$\widetilde{Gr}(\mathbb{D}^n) := \{C \subset \mathbb{D}^n | \dim C < \infty \ \dim C \equiv 0 \mod n\} \tag{2}$$

into which any $Gr_N(\mathbf{V})$ can be embedded by inclusion. (In Section 4, this set will be replaced by the infinite dimensional grassmannian $Gr(\mathbb{D}^n)$ which is its quotient by an equivalence relation.)

Note that the ring $\mathbb{C}[z]$ acts on $\mathbb{D}^n$ on the right. In particular, we define the composition $c \circ v$ for $c \in \mathbb{D}^n$ and $v(z) \in \mathbb{C}[z]$ by the formula

$$c \circ v \, (p_1(z), \ldots, p_n(z)) := c \, (v(z)p_1(z), \ldots, v(z)p_n(z)) \, . \tag{3}$$

Definition 1. *For $C \in \widetilde{Gr}(\mathbb{D}^n)$, let*

$$A_C := \{p(z) \in \mathbb{C}[z] | c \circ p \in C\} \tag{4}$$

denote the stabilizer of C in $\mathbb{C}[z]$.

Definition 2. *Given $c \in \mathbb{D}^n$ and $\lambda \in \mathbb{C}$ in the support of c, denote by $\mu_c(\lambda)$ the maximum non-negative integer $m \in \mathbb{N}$ such that some element of c written in the standard basis of $\mathbb{D}$ contains the expression $\delta_\lambda \circ \partial_z^{m-1}$ with non-zero coefficient. Note that $\mu_c(\lambda) > 0$ for all λ in the support of c. For convenience, we define $\mu_c(\lambda) = 0$ if λ is not in the support of c. Similarly, for a subspace $C \subset D_n$, let $\mu_C(\lambda)$ be the maximum over all $c \in C$ of $\mu_c(\lambda)$. Then, let $\sigma_C(z)$ be the polynomial*

$$\sigma_C(z) := \prod_{\lambda \in \mathbb{C}} (z - \lambda)^{\mu_C(\lambda)}. \tag{5}$$

Then, $\sigma_C(z)$ is a polynomial which is has a root at each λ of multiplicity sufficiently high that the following is true.

Lemma 1. *For any $c \in C$, the distribution $c \circ \sigma_C$ is the zero distribution*

$$c \circ \sigma_C \equiv 0. \tag{6}$$

Note: Thus, the ideal $\sigma_C(z)\mathbb{C}[z]$ is always contained in the ring A_C for all $C \in \widetilde{Gr}(\mathbb{D}^n)$.

The subspaces $C \in \widetilde{Gr}(\mathbb{D}^n)$ will be used below to perform Darboux transformations on arbitrary ordinary differential operators of order n.

3. Darboux Transformations

Let L_0 be a non-constant ordinary differential operator of order n. Without loss of generality, we suppose that L_0 is normalized to be in the form

$$L_0 = \partial^n + u_{n-2}(x)\partial^{n-2} + \cdots + u_0(x) \tag{7}$$

using the automorphisms of the ring of ordinary differential operators.

Let $\mathbf{F}$ be the kernel of the operator $L_0 - z$ for varying z. In particular, $f(x,z) \in \mathbf{F}$ satisfies the eigenvalue equation

$$L_0 f(x,z) = z f(x,z). \tag{8}$$

The space $\mathbf{F}$ has a unique basis of functions (viewed here as a vector) $\vec{f} := (f_1(x,z), \ldots, f_n(x,z))$ such that the Wronskian matrix $Wr(\vec{f})$ is the identity matrix when evaluated at $x = 0$. The functions $\vec{f}$ are analytic in the spectral parameter z. Then any point in $C \in Gr_N(\mathbb{D}^n)$ specifies a Darboux transformation of the ring $\mathbb{C}[L_0]$ as follows.

Lemma 2. *If $c \in \mathbb{D}^N$ is not the zero distribution, then $c(\vec{f}) \not\equiv 0$.*

Proof. Suppose $c \in \mathbb{D}^n$ satisfies $c(\vec{f}) = 0$. Then we must show that $c \equiv 0 \in \mathbb{D}^n$. Suppose that c is non-trivial and denote by λ_i ($1 \le i \le m$) the support of the distribution c and by μ_i the highest derivative evaluated at λ_i which appears in c. Let $\sigma(z)$ denote the polynomial

$$\sigma(z) = (z - \lambda_1)^{\mu_1} \prod_{i=2}^{m} (z - \lambda_i)^{\mu_i + 1}. \tag{9}$$

This polynomial is chosen specifically so that $c \circ \sigma$ is a distribution with support only at λ_1 which involves no derivatives higher than the 0^{th} derivative. In fact, since

$$0 = \sigma(L_0)c(\vec{f}) = c(\sigma(L_0)\vec{f}) \tag{10}$$

$$= c(\sigma(z)\vec{f}) = c \circ \sigma(\vec{f}) \tag{11}$$

we end up with an equation of the form

$$\sum_{j=1}^{n} \alpha_j f_j(x, \lambda_1) = 0 \tag{12}$$

where α_j is non-zero if and only if $d(\mu_1, j, \lambda_1)$ appears in a minimal representation of c. However, the functions $f_j(x, \lambda_1)$ are an independent set of functions spanning the kernel of $L_0 - \lambda_1$. In particular, all of the α_j must be zero, which contradicts the assumption that c involves the evaluation of some ${\mu_1}^{st}$ derivative at λ_1. □

Definition 3. *For any $C \in Gr_N(\mathbb{D}^n)$, it follows from Lemma 2 that the space spanned by the functions $c(\vec{f})$ for all $c \in C$ is an N dimensional space. Let the operator $K = K_C$ be the unique monic operator of order N such that the functions $c(\vec{f})$ are in the kernel of K. Picking a basis $C = \langle c_i \rangle$, one may compute K using the fact that*

$$Ku(x) = \frac{Wr(c_1(\vec{f}), \ldots, c_N(\vec{f}), u(x))}{Wr(c_1(\vec{f}), \ldots, c_N(\vec{f}))}. \tag{13}$$

Lemma 3. *Let Q be any ordinary differential operator such that* $\ker K \subset \ker Q$*, then $Q = LK$ for some ordinary differential operator L.*

Proof. We may always write $Q = LK + L'$ for some operator L' of order less than N. Then, since $\ker LK$ clearly contains the kernel of K, $\ker L'$ must also contain $\ker K$. However, the dimension of the kernel of a non-trivial operator is no greater than its order, and thus $L' = 0$. □

This now allows us to represent the ring A_C as a commutative algebra of ordinary differential operators through conjugation by K.

Claim 1. *For any $p \in A_C$, the operator*

$$L_p := Kp(L_0)K^{-1} \tag{14}$$

is an ordinary differential operator and thus the ring

$$\mathbf{R}_C := \{L_p | p \in A_C\} \tag{15}$$

is a commutative ring of ordinary differential operators of rank n.

Proof. First note that for $c \in C$, $Kp(L_0)$ satisfies

$$Kp(L_0)c(\vec{f}) = Kc(p(L_0)\vec{f}) \tag{16}$$

$$= Kc(p(z)\vec{f}) \tag{17}$$

$$= Kc \circ p(\vec{f}). \tag{18}$$

But then, since $c \circ p = c' \in C$ and $c'(\vec{f}) \in \ker K$ we have

$$Kp(L_0)c(\vec{f}) = 0. \tag{19}$$

By Lemma 3, this implies that $Kp(L_0) = LK$ for some operator L and then $L_p = L$.

That the rank of $\mathbf{R}_C$ is n follows from the fact that $\sigma_C(z)\mathbb{C}[z] \subset A_C$ and thus A_C contains a polynomial of every arbitrarily high degree. □

The transformation from $\mathbb{C}[L_0]$ to $\mathbf{R}_C$ is an iterated Darboux transformation [18] in the sense that it is achieved by N successive conjugations by linear operators of the form $Y = \partial - \frac{d}{dx}\log\phi(x)$ for eigenfunctions $\phi(x)$. In particular, in this case it is the iterated Darboux transformation by the N functions $c_i(\vec{f})$.

4. The Grassmannian $Gr(\mathbb{D}^n)$

Consider the linear map $\gamma : \mathbb{D}^n \to \mathbb{D}^n$ which takes c to $\gamma(c) = c \circ z$. Note that the kernel of this map is the n dimensional space

$$\ker\gamma = \langle(\delta_0, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, \delta_0)\rangle.$$

Then, one may consider the inverse map acting on subspaces

$$\gamma^{-1} : Gr(N, \mathbb{D}^n) \to Gr(N+n, \mathbb{D}^n).$$

Consequently, γ^{-1} is a map on the set $\widetilde{Gr}(\mathbb{D}^n)$.

Let $Gr(\mathbb{D}^n) = \widetilde{Gr}(\mathbb{D}^n)/\approx$ be the quotient by the equivalence relation induced by setting $C \approx \gamma^{-1}(C)$. To see that this has the structure of an infinite dimensional grassmannian, note that γ^{-1} embeds $Gr(N, \mathbb{D}^n)$ in $Gr(N+n, \mathbb{D}^n)$ and that $Gr(\mathbb{D}^n)$ is the direct limit as $N \to \infty$.

It is easy to check that this quotient agrees with the Darboux transformation construction introduced above. In particular, if $C' = \gamma^{-1}(C)$ (and thus C and C' represent the same equivalence class in $Gr(\mathbb{D}^n)$, then $\mathbf{R}_C$ and $\mathbf{R}_{C'}$ are the same ring of operators.

First let $c \in C$. By definition, this is equal to $c' \circ z$ for some $c' \in C'$. Now, take any $p \in A_{C'}$ and compute that $c \circ p = c' \circ z \circ p = c' \circ p \circ z = c_2' \circ z = c_2 \in C$. Thus $A_{C'} \subset A_C$. Similar arguments prove the converse.

Furthermore, one may check that $K_{C'} = K_C L_0$. (Note that $K_C L_0 \cdot c'(\vec{f}) = K_C \cdot c' \circ z(\vec{f}) = K_C \cdot c(\vec{f}) = 0$, and so it is the unique monic operator of order $n + r$ with the appropriate kernel.) Consequently

$$K_{C'} p(L_0) K_{C'}^{-1} = K_C p(L_0) K_C^{-1}.$$

In the remainder of the paper, I will abuse notation by referring to a vector space of distributions C in the grassmannian $Gr(\mathbb{D}^n)$. In such a situation, what is actually meant is any choice of representative of the class $[C] \in Gr(\mathbb{D}^n)$.

5. The Grassmannian Gr^n

Here we review relevant facts about the grassmannian Gr^n [20, 22] which is often used in the context of the KP hierarchy.

Consider the Hilbert space H^n of square-integrable vector valued functions $S^1 \to \mathbb{C}^n$, where $S^1 \subset \mathbb{C}$ is the unit circle. Denote by e_i for

$0 \leq i \leq n-1$ the n-vector which has the value 1 in the $i+1$ component and zero in the others. Aside from a shift in index, this is the standard basis of $\mathbb{C}^n$. This basis will be extended to the basis $\{e_i | i \in \mathbb{Z}\}$ of H^n for which $e_i = z^a e_b$ when $i = an + b$ for $0 \leq b \leq n-1$. The Hilbert space has the decomposition

$$H^n = H^n_+ \oplus H^n_- \tag{20}$$

where H^n_+ is the Hilbert closure of the subspace spanned by e_i for $i \geq 0$ and H^n_- is the Hilbert closure of the subspace spanned by e_i for $i < 0$.

Then Gr^n denotes the grassmannian of all closed subspaces $W \subset H^n$ such that the orthogonal projection $W \to H^n_-$ is a compact operator and such that the orthogonal projection $W \to H^n_+$ is Fredholm of index zero.

Definition 4. *Given a point W of the grassmannian, denote by A_W the ring*

$$A_W = \{f(z) = \sum_{i=-\infty}^{N} c_i z^i | N \in \mathbb{N},\ c_i \in \mathbb{C},\ fW \subset W\}. \tag{21}$$

In the case that A_W contains an element of every sufficiently large order N, the point W can be achieved as the L^2 boundary values of the section of a rank n holomorphic vector bundle over a complete irreducible complex curve X [20] after the removal of a specified smooth point x_∞ where a specified trivialization is used to identify sections of the bundle with $\mathbb{C}^r$ valued functions. In this case, the ring A_W contains the coordinate ring of the curve $X - \{x_\infty\}$ [20].

5.1. **The Dual Mapping.** We define the "dual map" by the formula

$$C \in \widetilde{Gr}(\mathbb{D}^n) \to W_C = z^{-N/n}\overline{V_C} \tag{22}$$

where $N = \dim C$,

$$V_C := \{\vec{p}(z) \in \mathbb{C}[z]^n | c(\vec{p}(z)) = 0 \qquad \forall c \in C\} \tag{23}$$

and the overline indicates Hilbert closure in H^n.

Note: By construction, this map is well defined on equivalence classes in $Gr(\mathbb{D}^n)$. In fact, if $C' = \gamma^{-1}(C)$ then $V_{C'} = zV_C$. Thus, since $\dim C' = n + \dim C$, $W_{C'} = W_C$. It will be shown below that the dual map embeds the grassmannian $Gr(\mathbb{D}^n)$ in the grassmannian Gr^n.

Claim 2. *The space W_C is in fact a point $W_C \in Gr^n$*

Proof. It is clear that $\pi_- : W_C \to H_-$ is compact and that all its elements are $L^2(S^1)$. The only thing to check is that $\dim\ker\pi_+ = \dim\operatorname{coker}\pi_+$. To see this, note that the projection map from $\overline{V_C}$ to

H_+ has N dimensional cokernel and 0 dimensional kernel. Then, multiplication by $z^{-N/n}$ shifts the index of the basis elements e_i back by exactly N. As a result, the dimension of the kernel and cokernel of $\pi_+ : W_C \to H_+$ are both

$$\dim(\bigoplus_{i=0}^{N-1} e_i \cap V_C). \tag{24}$$

□

Claim 3. *The map $C \to W_C$ embeds $Gr(\mathbb{D}^n)$ in Gr^n.*

Proof. Suppose $C_1, C_2 \in Gr(\mathbb{D}^n)$ get sent to the same point $W \in Gr^n$ via the dual map. Equivalence classes in $Gr(\mathbb{D}^n)$ contain elements of arbitrarily high dimension, and so we may suppose that C_1 and C_2 are chosen such that $\dim C_1 = \dim C_2 = N$. Then $V_{C_1} = V_{C_2} = z^{N/n}W \cap \mathbb{C}[z]$. But $\mathbb{D}^n$ embeds in the dual of $\mathbb{C}[z]^n$ and so the spaces C_i are identified by their kernels. □

Definition 5. *Let $Gr^n_{rat} \subset Gr_n$ denote the image of $Gr(\mathbb{D}^n)$ under the dual isomorphism. The relationship between the grassmannian $Gr(\mathbb{D}^n)$ and Gr^n_{rat} is that of dual grassmannians* [7]*. In particular, the infinite dimensional subspace W_C corresponds to its "perpendicular complement" C, which in this case is finite dimensional because W_C has finite codimension in $z^{-N/n}H^n_+$. This justifies the term "dual map" since it is indeed the classical dual isomorphism between $Gr(\mathbb{D}^n)$ and its image in Gr^n.*

Note that every point $W_C \in Gr^n_{rat}$ satisfies the condition

$$p(z)H^n_+ \subset W_C \subset z^{-j}H^n_+$$

for some positive integer j and polynomial $p(z)$. Thus, it is part of the vector generalization of the subgrassmannian Gr_1 described in [22] and the finitely supported used here are directly analogous to those used in that paper to find the τ-function for the N-soliton.

Claim 4. *In the special case that $W = W_C$ for some $C \in Gr(\mathbb{D}^n)$, the rings A_W (defined in Equation 21) and A_C (defined in Equation 4) are the same.*

Proof. Let $p(z) \in A_C$ and $\vec{q}(z) \in V_C$. Then $c(p(z)\vec{q}(z)) = c \circ p(\vec{q}) = 0$ because $c \circ p \in C$. Consequently, $p(z) \in A_W$ and so $A_C \subset A_W$.

Furthermore, note that $A_W \subset \mathbb{C}[z]$ since if $p(z) \in A_W$ it must satisfy $pV_C \subset V_C$. Now suppose $p(z) \in A_W$, then $p(z)\vec{q}(z) \in V_C$ for all $\vec{q} \in V_C$ which implies that $p(z)V_C \subset V_C$ and so, by duality $c \circ p \in C$ for all $c \in C$. □

6. KP Solutions

The vacuum operator L_0 can be given dependence upon the temporal parameters[1] $\mathbf{t} := (x, t_2, t_3, t_4, \ldots)$ so as to satisfy the n-KdV equations

$$\frac{\partial}{\partial t_i} L_0 = \left[\left(L_0^{i/n} \right)_+ , L_0 \right]. \tag{25}$$

Furthermore, we also assume that the eigenfunctions $f_j \in \mathbf{F}$ satisfy the equations

$$\frac{\partial}{\partial t_i} f_j(x, \mathbf{t}, z) = \left(L_0^{n/i} \right)_+ f_j(x, \mathbf{t}, z). \tag{26}$$

Given an n-KdV solution $L_0(x, \mathbf{t})$ and a basis of eigenfunctions $\vec{f}(x, \mathbf{t}, z) = (f_1, \ldots, f_n)$ as above, the paper [20] associates to each point $W \in Gr^n$ a solution $\mathbf{L}_W$ to the KP hierarchy, a vector of eigenfunctions $\vec{\psi}_W(x, z)$ and a commutative ring of ordinary differential operators $\mathbf{R}_W$ which commute with $\mathbf{L}_W$. The main result of this paper is that in the case $W = W_C$ for some $C \in Gr(\mathbb{D}^n)$, the KP solution can be simply determined as the iterated darboux transformation described above.

Definition 6. *Let $\Psi_0 := Wr(f_1, \ldots, f_n)$ denote the Wronskian matrix of the eigenfunctions f_i. Then, associated to a choice of $W \in Gr$ is the vector Baker function $\vec{\psi}_W(x, z)$* [20]*, which is the unique function such that*

a) $\vec{\psi}_W(x, z) = \left(\sum_{i=0}^{\infty} a_i(x) e_{-i} \right) \Psi_0$ *with* $a_0(x) \equiv 1$.
b) $\vec{\psi}_W(x, z) \in W$ *for all* x *in its domain.*

Lemma 4. *Let $C \in Gr(\mathbb{D}^n)$ and $W = W_C \in Gr^n$ be its image under the dual map. Then the vector Baker function of W can be computed as*

$$\vec{\psi}_W = z^{-N/n} K_C \vec{f} \tag{27}$$

Proof. Let $\vec{\phi}_C(x, z) := z^{-N/n} K_C \vec{f}$. It is sufficient to note that $\vec{\phi}_C(x, z)$ has the properties (a) and (b) from Definition 6. Define the vector $\vec{\alpha} := (\alpha_1, \ldots, \alpha_n)$ by the formula

$$K f_i(x, z) = \sum_{j=0}^{n-1} \alpha_j(x, z) \left(\frac{\partial}{\partial x} \right)^j f_i(x, z). \tag{28}$$

[1] It is convenient to identify the spatial parameter x with the temporal parameter t_1.

Then note that $\vec{\phi}_C = z^{-N/n}\vec{\alpha}\Phi_0$. However, it is clear that $z^{-N/n}\vec{\alpha}$ is of the form $\sum a_i(x)e_{-i}$. In fact, this is true merely because K is a monic operator of order N and so $\alpha_0(x,z)$ is a monic polynomial in z of degree N/n and each α_j for $j>0$ is a polynomial of lower degree. This shows that $\vec{\phi}_C$ satisfies property (a). Furthermore, note that $\vec{g}(z)\in W_C$ if and only if

$$c(z^{N/n}\vec{g}(z)) = 0 \qquad \forall c\in C. \tag{29}$$

However, it then follows that $\vec{\phi}_C$ satisfies property (b) as well since for $\vec{g}=\phi_C$

$$\begin{aligned} c(z^{N/n}\vec{g}) &= c(K\vec{f}) && (30)\\ &= Kc(\vec{f}) && (31)\\ &= 0 && (32)\end{aligned}$$

for all $c\in C$ by definition of K. □

Definition 7. *(Previato-Wilson* [20]*) Given $W\in Gr^n$, let S be the unique 0^{th} order pseudo-differential operator such that $\vec{\psi}=S\vec{f}$ (where application of pseudo-differential operators to $\vec{f}$ is defined so that $L_0^{-1}\vec{f}=z^{-1}\vec{f}$). Then $\mathbf{L}_W = SL_0^{1/n}S^{-1}$ is a solution to the KP hierarchy.*

Lemma 5. *For $C\in Gr(\mathbb{D}^n)$ and $W=W_C\in Gr^n$, the operators $K=K_C$ and $S=S_W$ are related by the formula*

$$S = K(L_0)^{-N/n}.$$

Proof. The pseudo-differential operator $L_0^{-N/n}$ has order $-N$ and K has order N. Consequently, $K(L_0)^{-N/n}$ has order 0. Furthermore, by definition $L_0^{-1}\vec{f}=z^{-1}\vec{f}$. Therefore,

$$K(L_0)^{-N/n}\vec{f} = z^{-N/n}K\vec{f}$$

which, by Lemma 4, is the vector Baker function. □

Theorem 1. *If L_0 satisfies the n-KdV hierarchy* (25) *and $\vec{f}$ satisfies* (26)*, then for any $C\in Gr(\mathbb{D}^n)$, the pseudo-differential operator*

$$\mathbf{L}_C = K_C L_0^{1/n} K_C^{-1}$$

is a solution to the KP hierarchy. Furthermore, letting $W=W_C\in Gr^n$ be the image of C under the dual isomorphism, then $\mathbf{L}_C=\mathbf{L}_W$, the KP solution associated to (W,L_0) by the Krichever-Novikov construction [12, 13] *as described in* [20].

Proof. By definition, $\mathbf{L}_W = SL_0^{1/n}S^{-1}$, but we have by Lemma 5 that this is $K(L_0)^{-N/n}L_0^{1/n}L_0^{N/n}K^{-1} = KL_0^{1/N}K^{-1} = \mathbf{L}_C$. Then, since $\mathbf{L}_W$ is a solution to the KP hierarchy, so is $\mathbf{L}_C$. □

7. Tau Functions

The present section demonstrates that a Wronskian formula for τ-functions analogous to the one utilized in [25] holds in Gr^n_{rat}. It will first be necessary to define precisely what is meant by a τ-function in this context since τ-functions were not discussed in [20]. Consequently, the following can be seen as the definition of τ-functions for the grassmannian construction described by Previato and Wilson in [20].

As you will see, it is merely the standard definition applied to Gr^n with time dependence determined by the matrix Ψ_0. However, the definition raises some interesting questions which will not be addressed in the present work. The established theory of τ-functions is quite deep, relating topics such as bilinear forms of integrable systems [6], the Plücker coordinates of infinite dimensional grassmannian manifolds [21, 22], representation theory of infinite dimensional Lie algebras [3, 8], and "bosonization" (the manifestation of a fermion in a bosonic field) [17, 24]. Although the relationship of the τ-functions defined below to the the first two topics will be discussed, it is not clear how (or if) they relate to the representation of Lie algebras or the boson-fermion correspondence.

7.1. General Definitions.

A Hilbert basis for the point $W \in Gr^n$ is said to be an *admissible basis* [19, 22] if its projection to the standard basis $\{e_i\}$ for H^n_+ differs from the identity by an operator of trace class [23]. It is convenient to identify an admissible basis $\{w_i\}$ with the linear map

$$\begin{aligned} w : H^n_+ &\to W \\ e_i &\mapsto w_i. \end{aligned}$$

The *frame bundle* of Gr^n is the set of pairs (W, w) where $W \in Gr^n$ and $w : H^n_+ \to W$ is an admissible basis in the sense discussed above.

Let § be the set of sequences $(s_0, s_1, \cdots)$ such that $s_i < s_{i+1}$ and $s_i = i$ for i sufficiently large. Then given some $S \in §$, the subspace $W_S \subset H^n$ spanned by e_{s_i} is a point of the grassmannian (in fact, it is the center of a Schubert cell) and $\{e_{s_i}\}$ is an admissible basis. For example, $S_+ := (0, 1, 2, \cdots)$ generates the standard basis for H^n_+.

Definition 8. *Let Λ denote the infinite alternating exterior algebra generated by the alternating tensors*

$$e_{s_0} \wedge e_{s_1} \wedge \cdots$$

for all $S \in \S$. Given an admissible basis determined by the map w, we use the notation $|w\rangle$ to denote the alternating tensor

$$|w\rangle = w_0 \wedge w_1 \wedge \cdots \in \Lambda.$$

Note, in particular, that if w and w' are two admissible bases for W, then $|w\rangle = \lambda |w'\rangle$ for some constant $\lambda \neq 0$. Consequently, $|W\rangle$ is a well defined element of $\mathbb{P}\Lambda$ (namely, the class of $|w\rangle$ for any admissible basis w of W). Thus, this procedure embeds the grassmannian in a projective space. The Plücker coordinates are indexed by the set $\S$ and are computed as the determinant projection map to W_S with w [19]. Furthermore, denoting by $\langle S|W\rangle$ the Plücker coordinate corresponding to $S \in \S$ for the point $W \in Gr^n$, one can expand $|W\rangle$ in a formal sum

$$|W\rangle = \sum_{S \in \S} \langle S|W\rangle e_{s_0} \wedge e_{s_1} \wedge e_{s_2} \wedge \cdots$$

For notational convenience, let

Of course, each Plücker coordinate by itself gives very little information, since they are only projectively defined. All one can say about a single coordinate is whether it is zero or not. Since $\langle S_+|W\rangle$ can also be computed as the determinant of the projection map from W to H^n_+, it is clear that the following holds: $\langle S_+|W\rangle = 0$ if and only if W is the graph of some function $F_W : H^n_+ \to H^n_-$. This is generically true and the set of points such that $\langle S_+|W\rangle \neq 0$ is standardly referred to as the *big cell*.

To define τ_W requires an operator A acting on the frame bundle. As described in [19], an operator on the frame bundle is a pair (g, q) where $g \in GL(H^n)$ which has the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{33}$$

relative to the splitting $H^n = H^n_+ \oplus H^n_-$ and $q : H^n_+ \to H^n_+$ such that aq^{-1} differs from the identity by an operator of trace class. The action on the frame bundle is given by

$$A : (W, w) \to (gW, gwq^{-1}).$$

Note: In the case that g is invertible and $c = 0$ (where c is as in (33)), one may always choose $q = a$. Thus, in such a case, it is always sufficient to merely specify g. This will be the case in all examples that follow.

We can also let A act on $|w\rangle$ by defining

$$|A|w\rangle = |gwq^{-1}\rangle.$$

Finally, letting A depend on time parameters $\mathbf{t} = (t_1, t_2, \ldots)$ we can define

$$\tau_W(\mathbf{t}) = \langle S_+|A|W\rangle \tag{34}$$

making τ a projectively defined function of the time variables. (You can view $A(\mathbf{t})$ as a "connection" allowing one to compare the first Plücker coordinate at different points in the orbit $\{g(\mathbf{t})W\}$.)

In different contexts, different choices of A are appropriate. For example, to construct τ-functions of the KP hierarchy [22], let $n = 1$ and $A = (g, a)$ where $g = exp(\sum t_i z^i)$.

Definition 9. *The τ-function for the to the point $W \in Gr^n$ (also with L_0 and Ψ_0 fixed as above) is given by* (34) *with $A(\mathbf{t}) = (\Psi_0^{-1}(\mathbf{t}), a)$.*

This definition having been made, it now remains to demonstrate that the definition is useful. In particular, it remains to show that τ_W provides information about the KP solution associated to W and L_0. The following subsection will determine τ_W as a Wronskian determinant in the case of $W \in Gr^n_{rat}$. As an application, it is shown how one is able to determine solutions to the KP equation using τ_W.

7.2. **The Case of $W \in Gr^n_{rat}$.** The major result of this section is that for $W = W_C$ for some $C \in \mathbb{D}^n$, the corresponding τ-function is easily computed in terms of the distribution space C.

Theorem 2. *Suppose the point $W \in Gr^n_{rat}$ is given by the mn dimensional subspace $C \subset \mathbb{D}^n$, and $\{c_i\}$ is any basis of C, then τ_W can be determined as the Wronskian determinant*

$$\tau_W = \det\left(\frac{d^i}{dx^i} c_j(f_1, \ldots, f_r)\right) \qquad 1 \le i, j \le mn.$$

Proof. Without loss of generality, it will be assumed that W is in the big cell. This is sufficient since the big cell is dense in Gr^r and so the general case follows by continuity.

Then one may choose for the map w generating an admissible basis of W the map of the form

$$w = \begin{pmatrix} I \\ F_W \end{pmatrix}$$

for the operator $F_W : H^n_+ \to H^n_-$ whose graph is W. If Ψ_0^{-1} has the form

$$\Psi_0^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with respect to the splitting $H^n = H^n_+ \oplus H^n_-$ then $\tau_W(\mathbf{t})$ can be found as the determinant

$$\begin{aligned}\tau_W &= \det(I + bF_W a^{-1}) \\ &= \det(a^{-1})\det(I + bF_W a^{-1})\det(a) \\ &= \det(I + a^{-1}bF_W).\end{aligned}$$

Let $\alpha_i : H^n_+ \to \mathbb{C}$ for $1 \leq i \leq mn$ be defined by the property that

$$\vec{f} + \sum_{i=1}^{mn} \alpha_i(\vec{f})e_{-i} \in W \qquad \forall \vec{f} \in H^n_+.$$

Furthermore, define $\mathbf{E}_i = a^{-1}be_{-i}$ for $1 \leq i \leq rN$. By the functorial isomorphism between $H^{n*}_+ \otimes H^n_+$ and $Hom(H^n_+, H^n_+)$ [14], we can view the map $a^{-1}bF_W : H^n_+ \to H^n_+$ as the sum $\sum_{i=1}^{mn} \alpha_i \otimes \mathbf{E}_i$. Thus

$$\tau_W = \det(I + \sum_{i=1}^{mn} \alpha_i \otimes \mathbf{E}_i).$$

Then, as one can see by rewriting this matrix in terms of any basis including the elements $\mathbf{E}_i$, the determinant on the right is equal to the determinant of the $mn \times mn$ matrix [22] T defined by

$$T_{ij} = \delta_{ij} + \alpha_i(\mathbf{E}_j).$$

Note that by definition,

$$c(z^m\vec{f} + \sum_{i=1}^{mn} \alpha_i(\vec{f})e_{mn-i}) = 0$$

for all $c \in C$ and $\vec{f} \in H^n_+$. Thus, in particular, we have

$$c_j(z^m\vec{f}) = -\sum_{i=1}^{mn} c_j(e_{mn-i}\alpha_i(\vec{f})) \tag{35}$$

for the each basis element $1 \leq j \leq mn$. Therefore,

$$\begin{pmatrix} c_1(z^m\vec{f}) \\ \vdots \\ c_{mn}(z^m\vec{f}) \end{pmatrix} = M \begin{pmatrix} \alpha_1(\vec{f}) \\ \vdots \\ \alpha_{mn}(\vec{f}) \end{pmatrix}$$

where M is the $mn \times mn$ matrix with elements $M_{ij} = -c_j(e_{mn-i})$.

Then, by factoring out the matrix[2] M^{-1} from the matrix T, we find [3]

$$
\begin{aligned}
\tau_W &= \det(T) &(36)\\
&= \det(M^{-1})\det(M_{ij}+c_j(\mathbf{E}_i z^N)) &(37)\\
&= \det(M^{-1})\det(c_j(z^N(a^{-1}b+I)e_{-i})) &(38)\\
&= \det(M^{-1})\det(c_j(z^n(-e_{-i}\Psi_0^{-1})_-\Psi_0)) &(39)\\
&= \det(M^{-1})\det(c_j(\frac{\partial^i}{dx^i}(f_1,\ldots,f_n))). &(40)
\end{aligned}
$$

Then, since τ_W is defined only up to a constant multiple in any case, the constant $\det(M^{-1})$ can be ignored and the theorem is proved. □

Note: The Wronskian formula above was used in [10] to investigate bispectral algebras of ordinary differential operators. Consequently, an application for these τ-functions has already been found which is unrelated to the dynamical aspects of soliton equations. In the next section, the role of these τ-functions in the KP theory will be elucidated allowing for the construction of new KP solutions.

7.2.1. *τ and Darboux Transformations.* It is known that the iterated Darboux transformation of the KP solutions corresponding to τ_0 by the eigenfunctions ϕ_j has the effect

$$\tau_0 \mapsto (\tau_0)(Wr(\phi_j))$$

on the τ-function [1, 18]. (That is, it multiplies τ by the Wronskian of the eigenfunctions.)

Since it was shown above that the operator $\mathbf{L}_W$ corresponding to a point $W \in Gr^n_{rat}$ is the iterated darboux transformation of L_0 by the eigenfunctions

$$\phi_j = c_j(f_1,\ldots,f_n),$$

one may conclude that $\tau_{\mathbf{L}_W}$ is the product of τ_{L_0} with the Wronskian of these functions.[4] Furthermore, we found above that this wronskian is τ_W and so we have the following result:

[2]Considering the equations (35) as a system of mn equations to determine the mn "unknowns", α_i, it becomes clear that the matrix M^{-1} exists if and only if α_i exist. Consequently, the matrix M is invertible because W is in the big cell.

[3]The notation $(\vec{f})_-$ in step (39) above denotes projection onto H^n_- and moving from (39) to step (40) is achieved by the substitution $z^{jn}f_i \to L_0^j f_i$ to remove factors of z, followed by multiplication by a matrix of determinant 1 written in terms of the coefficients in the power series representation of Ψ_0^{-1} around $z=0$.

[4]Here $\tau_{\mathbf{L}_W}$ and τ_{L_0} denote appropriate gauges of the τ-functions corresponding to the solutions $\mathbf{L}_W$ and $L_0^{1/n}$ of the KP hierarchy.

Claim 5. *The τ-function corresponding to $\mathbf{L}_W$ is*

$$\tau_{\mathbf{L}_W} = \tau_{L_0}\tau_W$$

or alternatively, τ_W is a quotient of τ-functions

$$\tau_W = \frac{\tau_{\mathbf{L}_W}}{\tau_{L_0}}.$$

It it then clear that τ_W is not itself a τ-function of the KP hierarchy in the usual sense. In particular, we have seen that the product $\tau_{L_0}\tau_W$ satisfies the Hirota bilinear equations of the KP hierarchy. So, unless $\tau_{L_0} = e^{\alpha t_1+\beta}$, it is not the case that τ_W is also a solution of the equations. However, it might be profitable to consider the functions τ_W as solutions to the "non-autonomous" bilinear equation given by first multiplying by τ_{L_0} and then applying the bilinear operators.

7.3. **Example.** The Airy vacuum is the operator

$$L_0 = \partial^2 - \frac{2x}{3t+c}$$

whose normalized vector of eigenfunctions $\vec{f} = (f_1, f_2)$ can be written simply in terms of the classical Airy functions $Ai(z)$ and $Bi(z)$ (see [9] for details). For any value of $c \in \mathbb{C}$, this operator is a solution of the first two equations of the KdV hierarchy. (Dependence upon time variables t_i for $i > 3$ will not be considered in these examples for the sake of simplicity.)

Consider, for example,

$$c_1(f,g) = f(0) + f(2) \qquad c_2(f,g) = g(0).$$

The τ-function of the corresponding point

$$W_1 = \{(1-z^{-1},0),(0,1),(z-2,0),(0,z),(z^n(z-2),0),(0,z^n),\ldots\}$$

can be computed as the wronskian determinant of $\{c_1(\vec{f}), c_2(\vec{f})\}$. In the case of vacuum $L_0 = \partial^2 - \frac{2x}{3t}$ this turns out to be

$$\begin{aligned}\tau_{W_1} &= 1 + \frac{-e^{2y}\Gamma(1/3)\Gamma(2/3)}{4}(3Ai[\theta(3t+x)]Ai'[\theta x] - 3Ai[\theta x]Ai'[\theta(3t+x)]\\ &\quad +\sqrt{3}Ai'[\theta(3t+x)]Bi[\theta x] + \sqrt{3}Ai'[\theta x]Bi[\theta(3t+x)]\\ &\quad -\sqrt{3}Ai[\theta(3t+x)]Bi'[\theta x] - Bi[\theta(3t+x)]Bi'[\theta x]\\ &\quad -\sqrt{3}Ai[\theta x]Bi'[\theta(3t+x)] + Bi[\theta x]Bi'[\theta(3t+x)])\end{aligned}$$

for $\theta = (\frac{2}{3t})^{1/3}$. Then $u(x,y,t) = \frac{2x}{3t+c} + \frac{\partial^2}{\partial x^2}\log\tau_W$ is a non-rational solution to the KP equation whose geometric spectral data are a rational curve with a node (given by identifying the points $z = 2$ and $z = 0$) and a rank two bundle.

Next, consider the two dimensional dual space with basis

$$c_1(f,g) = f'(0) + kf(0) \qquad c_2(f,g) = g'(0) + \frac{c}{2}f(0) + kg(0).$$

(These conditions clearly place a cusp at the point $z = 0$ of the rational spectral curve.) Under the dual mapping, this corresponds to the point

$$W = \{(-k + z^{-1}, -c/2), (0, ck - cz^{-1}), (z, 0), (0, z), \ldots\}.$$

These coordinates were chosen specifically to generate a rational solution when used in conjunction with the Airy vacuum. In particular, one finds that the corresponding τ-function is

$$\tau_W = \left(x(3t + c) - 2(y + k)^2\right)$$

The KP solution corresponding to this τ-function is

$$u(x, y, t) = -\frac{2(3t + c)^2}{((3t + c)x - 2(y + k)^2)^2} - \frac{2x}{3t + c}$$

which was found through alternative means by Grünbaum [5]. Note that the parameters k and c determine the y and t flow respectively and, in particular, that the coefficients of the dual elements are linear functions of these KP flow parameters.

8. Spectral Geometry and True Rank

Since the commutative ring of ordinary differential operators $\mathbf{R}_C$ is isomorphic to the subring $A_C \subset \mathbb{C}[z]$ for all values of the temporal parameters, it is clear both that they induce an isospectral deformation of the ring and furthermore that the ring is a (singular) rational curve. In particular, since A_C is determined from $\mathbb{C}[z]$ through differential conditions at the support of the elements of C, the curve is constructed from $\mathbb{P}^1$ with local parameter z by introducing singularities (both cuspidal and nodal) at $z = \lambda_i$ for all λ_i in the support of C.

The solutions to the KP hierarchy corresponding to higher rank geometric spectral data are of particular interest [12, 13, 20]. Although all rings $\mathbf{R}_C$ generated by the construction above are commutative rings of rank n, and therefore have spectral data consisting of a rank n sheaf over their spectral curves, we would not like to consider all of the solutions $\mathbf{L}_C$ to the KP hierarchy as "rank n" solutions. In particular, if $\mathbf{R}_C$ is (resp. is not) contained in some larger commutative ring of lower rank, one refers to $\mathbf{L}_C$ as having "fake rank n" (resp. true rank n). If we define the true rank of an ordinary differential operator to be the rank of its centralizer, then the following lemma (which also appears in [10], but is repeated here for clarity) demonstrates that rank is preserved by Darboux transformation.

Lemma 6. *If* $X = Y_1Y_2$ *and* $\hat{X} = Y_2Y_1$ *then* X *and* $\hat{X}$ *have the same true rank.*

Proof. If Q is an operator commuting with X, then

$$0 = Y_2[Q,X]Y_1 \tag{41}$$
$$= Y_2QY_1Y_2Y_1 - Y_2Y_1Y_2QY_1 \tag{42}$$
$$= [Y_2QY_1, \hat{X}]. \tag{43}$$

Let r be the true rank of $\hat{X}$. Then, by (43), we have $ord(Y_2QY_1) \equiv 0 \bmod r$. But, $ord(Y_2QY_1) = ord(Y_2Y_1) + ord(Q)$ and since $ord(Y_2Y_1) \equiv 0 \bmod r$ we conclude that $ord(Q) \equiv 0 \bmod r$. Therefore, the true rank of $\hat{X}$ divides the true rank of X. Then, by symmetry, the true ranks are equal. □

As a consequence, we have the following result:

Theorem 3. *The ring* $\mathbf{R}_C$ *has true rank* n *iff the centralizer of the vacuum operator* L_0 *is* $\mathbb{C}[L_0]$.

In particular, the Airy vacuum used in the examples above is of true rank two [15] and therefore the KP solutions constructed there are of true rank two as well.

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