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Spectral difference equations satisfied by KP soliton wavefunctions

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Abstract. It is by now well known that the wavefunctions of rational solutions to the KP hierarchy which can be achieved as limits of the pure n-soliton solutions satisfy an eigenvalue equation for ordinary differential operators in the spectral parameter. This property is known as 'bispectrality' and has proved to be both interesting and useful. In this paper, it is shown that all pure soliton solutions of the KP hierarchy (as well as their rational degenerations) satisfy an eigenvalue equation for a non-local operator constructed by composing ordinary differential operators in the spectral parameter with *translation* operators in the spectral parameter, and therefore have a form of bispectrality as well.

1. Introduction

1.1. The KP hierarchy and bispectrality

Let $\mathbb{D}$ be the vector space spanned over $\mathbb{C}$ by the set

$$\{\Delta(n,\lambda) \mid \lambda \in \mathbb{C}, n \in \mathbb{N}\}$$

whose elements differentiate and evaluate functions of the variable z:

$$\Delta(n,\lambda)[f(z)] := f^{(n)}(\lambda).$$

The elements of $\mathbb{D}$ are thus *finitely supported distributions* on appropriate spaces of functions in z. For lack of a better term, we shall continue to call them distributions even though their main use in this paper will be their application to functions of two variables. (Such distributions were called 'conditions' in [20] since a KP wavefunction was specified by requiring that it be in their kernel.) Note that if $c \in \mathbb{D}$ and $f(x,z)$ is sufficiently differentiable in z on the support of c, then $\hat{f}(x) = c[f(x,z)]$ is a function of x alone. Furthermore, note that one may 'compose' a distribution with a function of z, i.e. given $c \in \mathbb{D}$ and $f(z)$ (sufficiently differentiable on the support of c) then there exists a $c' := c \circ f \in \mathbb{D}$ such that

$$c'(g(z)) = c(f(z)g(z)) \qquad \forall g.$$

The subspaces of $\mathbb{D}$ can be used to generate solutions to the KP hierarchy [18] in the following way. Let $C \subset \mathbb{D}$ be an n-dimensional subspace with basis $\{c_1, \ldots, c_n\}$. Then, if $K = K_C$ is the unique, monic ordinary differential operator in x of order n having the functions $c_i(\mathrm{e}^{xz})$ in its kernel (see (3)) we define $\mathcal{L}_C = K\frac{\partial}{\partial x}K^{-1}$ and $\psi_C = \frac{1}{z^n}K\mathrm{e}^{xz}$. The connection to integrable systems comes from the fact that adding dependence to C on a sequence of variables t_j $(j = 1, 2, \ldots)$ by letting $C(t_j)$ be the space with basis

$$\{c_1 \circ \mathrm{e}^{\sum t_j z^j}, c_2 \circ \mathrm{e}^{\sum t_j z^j}, \ldots, c_n \circ \mathrm{e}^{\sum t_j z^j}\}$$

it follows that the 'time dependent' pseudo-differential operator $\mathcal{L} = \mathcal{L}(t_j)$ satisfies the equations of the KP hierarchy [2, 10, 11, 20]

$$\frac{\partial}{\partial t_j}\mathcal{L} = [(\mathcal{L}^j)_+, \mathcal{L}].$$

The *wavefunction* $\psi_C(x, z)$ generates the corresponding subspace of the infinite-dimensional Grassmannian Gr [18] which parametrizes KP solutions and thus it is not difficult to see that this construction produces precisely those solutions associated to the sub-Grassmannian $\mathrm{Gr}_1 \subset \mathrm{Gr}$ [18, 20].

Moreover, the ring $A_C = \{p \in \mathbb{C}[z] | c_i \circ p \in C, 1 \leqslant i \leqslant n\}$ is necessarily non-trivial (i.e. contains non-constant polynomials) and the operator $L_p = p(\mathcal{L})$ is an *ordinary* differential operator for every $p \in A_C$ and satisfies

$$L_p \psi_C(x, z) = p(z)\psi_C(x, z). \tag{1}$$

The subject of this paper is the existence of *additional* eigenvalue equations satisfied by $\psi_C(x, z)$. In particular, we wish to consider the question of whether there exists an operator $\hat{\Lambda}$ acting on functions of the variable z such that

$$\hat{\Lambda}\psi_C(x, z) = \pi(x)\psi_C(x, z) \tag{2}$$

where $\pi(x)$ is a non-constant function of x. For example, the following theorem is due to Wilson in [20].

Theorem 1.1. *In addition to (1) the wavefunction $\psi_C(x, z)$ is also an eigenfunction for a ring of ordinary differential operators in z with eigenvalues depending polynomially on x if and only if C has a basis of distributions each of which is supported only at one point.*

In other words, for this special class of KP solutions for which the coefficients of $\mathcal{L}$ are rational functions of x, the wavefunction ψ_C satisfies an additional eigenvalue equation of the form (2) where $\hat{\Lambda}$ is an ordinary differential operator in z and $\pi(x)$ a non-constant polynomial in x†. Together (1) and (2) are an example of *bispectrality* [3, 5, 8]. The bispectral property is already known to be connected to other questions of physical significance such as the time-band limiting problem in tomography [7], Huygens' principle of wave propagation [4], quantum integrability [9, 19] and, especially in the case described above, the self-duality of the Calogero–Moser particle system [11, 20, 21].

It is known that the only subspaces C for which the corresponding wavefunction satisfies (1) and (2) with L_p and $\hat{\Lambda}$ ordinary differential operators in x and z respectively are those described in theorem 1.1. However, suppose we allow $\hat{\Lambda}$ to involve not only differentiation and multiplication in z but also *translation* in z and call this more *general* situation t-bispectrality‡. It will be shown below that there are more KP solutions which are bispectral in this sense. In particular, it will be shown that the KP solution associated to *any* subspace C shares its eigenfunction with a ring of translational-differential operators in the spectral parameter.

† Moreover, he demonstrated that up to conjugation or change of variables, the operators L_p found in this way are the only bispectral operators which commute with differential operators of relatively prime order, but this fact will not play an important role in this paper.

‡ It should be noted that the term 'bispectrality' already applies to more general situations than simply differential operators [8], but in the case of the KP hierarchy I believe only differential bispectrality has thus far been considered.

1.2. Notation

Using the shorthand notation $\partial = \frac{\partial}{\partial x}$ any ordinary differential operator in x can be written as

$$L = \sum_{i=0}^{N} f_i(x)\partial^i \qquad N \in \mathbb{N}.$$

We say that a function of the form

$$f(x) = \sum_{i=1}^{n} p_i(x)\mathrm{e}^{\lambda_i x} \qquad \lambda_i \in \mathbb{C},\ p_i \in \mathbb{C}[x]$$

is a *polynomial-exponential function* and that the quotient of two such functions is *rational-exponential*. This paper is especially concerned with the ring of differential operators with rational-exponential coefficients and especially with the subring having polynomial-exponential coefficients. Similarly, we will write $\partial_z = \frac{\partial}{\partial z}$ but will need to consider only differential operators in z with rational coefficients.

For any $\lambda \in \mathbb{C}$ let $\boldsymbol{S}_\lambda = \mathrm{e}^{\lambda\partial_z}$ be the translational operator acting on functions of z as

$$\boldsymbol{S}_\lambda[f(z)] = f(z+\lambda).$$

Then consider the ring of translational differential operators $\mathbb{T}$ generated by these translational operators and ordinary differential operators in z. Any translational differential operator $\hat{T} \in \mathbb{T}$ can be written as

$$\hat{T} = \sum_{i=1}^{N} p_i(z, \partial_z)\boldsymbol{S}_{\lambda_i}$$

where p_i are ordinary differential operators in z with rational coefficients and $N \in \mathbb{N}$. Note that the ring of ordinary differential operators in z with rational coefficients is simply the subring of $\mathbb{T}$ of all elements which can be written as $p\boldsymbol{S}_0$ for a differential operator p.

2. Translational bispectrality of $\mathbb{C}[\partial]$

It has been frequently observed that the ring $\mathcal{A} = \mathbb{C}[\partial]$ of constant coefficient differential operators in x is *bispectral* since it has the eigenfunction e^{xz} which it shares with the ring of constant coefficient differential operators in z. Here, however, we will consider a more general form of bispectrality for the ring $\mathcal{A}$.

Let $\mathcal{A}' \subset \mathbb{T}$ be the ring of constant coefficicient *translational* differential operators. Note that for any element $\hat{T} \in \mathcal{A}'$ of the form

$$\hat{T} = \sum_{i=1}^{N} p_i(\partial_z)\boldsymbol{S}_{\lambda_i}$$

one has simply that

$$\hat{T}[\mathrm{e}^{xz}] = \left(\sum_{i=1}^{N} p_i(x)\mathrm{e}^{\lambda_i x}\right)\mathrm{e}^{xz}.$$

In particular, e^{xz} is an eigenfunction for the operator with an eigenvalue which is a polynomial-exponential function of x. Consequently, the rings $\mathcal{A}$ and $\mathcal{A}'$ are both bispectral, sharing the common eigenfunction e^{xz}.

Let $\mathcal{R}$ be the ring of differential operators in x with polynomial-exponential coefficients and $\mathcal{R}'$ be the ring of translational-differential operators in z with rational coefficients. Note that $\mathcal{R}$ is generated by $\mathcal{A}$ and the eigenvalues of the operators in $\mathcal{A}'$ while $\mathcal{R}'$ is generated by $\mathcal{A}'$ and the eigenvalues of the elements of $\mathcal{A}$. It then follows [1] (see also [13]) that the map $b : \mathcal{R} \to \mathcal{R}'$ defined by the relationship

$$L[\mathrm{e}^{xz}] = b(L)[\mathrm{e}^{xz}] \qquad \forall L \in \mathcal{R}$$

is an anti-isomorphism of the two rings.

3. Translational bispectrality of KP solitons

Let us say that a finite-dimensional subspace $C \subset \mathbb{D}$ is *t-bispectral* if there exists a translational-differential operator $\hat{\Lambda} \in \mathbb{T}$ satisfying equation (2) for the corresponding KP wavefunction $\psi_C(x, z)$. By theorem 1.1 and the fact that the ring of rational coefficient ordinary differential operators in z is contained in $\mathbb{T}$, we know that C is t-bispectral† if it has a basis of point-supported distributions. Here we shall show that, in fact, all subspaces $C \subset \mathbb{D}$ are t-bispectral.

An important object in much of the literature on integrable systems is the 'tau function'. The tau function of the KP solution associated to C can be written easily in terms of the basis $\{c_i\}$. In particular, define (cf [20])

$$\tau_C(x) = \mathrm{Wr}(c_1(\mathrm{e}^{xz}), c_2(\mathrm{e}^{xz}), \ldots, c_n(\mathrm{e}^{xz}))$$

to be the Wronskian determinant of the functions $c_i(\mathrm{e}^{xz})$. Similarly, there is a Wronskian formula for the coefficients of the operator K_C since its action on an arbitrary function $f(x)$ is given as

$$K_C(f(x)) = \frac{1}{\tau_C(x)} \mathrm{Wr}(c_1(\mathrm{e}^{xz}), c_2(\mathrm{e}^{xz}), \ldots, c_n(\mathrm{e}^{xz}), f(x)). \tag{3}$$

Then the coefficients of the differential operator $\bar{K}_C := \tau_C(x) K_C(x, \partial)$ are all polynomial exponential functions.

Lemma 3.1. *For any $C \subset \mathbb{D}$ there exists a constant coefficient operator $L_0 \in \mathcal{A}$ which factors as*

$$L_0 = \bar{Q}_g \circ \frac{1}{\pi(x)} \circ \bar{K}_C$$

where $\bar{Q}_g, \bar{K}_C \in \mathcal{R}$ and $\pi(x) = g(x)\tau_C(x) \in \mathcal{R}$ is a polynomial-exponential function.

Proof. Let $\lambda_i \in \mathbb{C}$ $(1 \leqslant i \leqslant N)$ be the support of the distributions in C and m_i be the highest derivative taken at λ_i by any element of C. Then the polynomial

$$q_C(z) := (z - \lambda_i)^{m_i+1} \tag{4}$$

has the property that $c \circ q_C \equiv 0$ for any $c \in C$. Let $L_0 := q_C(\partial)$ and consider $L_0[c(\mathrm{e}^{xz})]$ for any element $c \in C$. Since L_0 is an operator in x alone, it commutes with c and we have

$$L_0[c(\mathrm{e}^{xz})] = c(L_0[\mathrm{e}^{xz}]) = c(q(z)\mathrm{e}^{xz}) = c \circ q(\mathrm{e}^{xz}) = 0.$$

So, by the definition of K_C, we see that L_0 annihilates the kernel of K_C and thus has a right factor of K_C. This gives a factorization of the form $L_0 = Q \circ K_C$ with Q having rational exponential coefficients. Then, by choosing a polynomial exponential function $g(x)$ so that $\bar{Q}_g := Q \circ g(x) \in \mathcal{R}$ we find the desired factorization. □

Given this factorization, the t-bispectrality of all C's now follows from theorem 4.2 in [1].

† . . . and also bispectral in the sense of [20].

Theorem 3.1. *For any subspace $C \subset \mathbb{D}$ the corresponding KP wavefunction $\psi_C(x,z)$ satisfies the eigenvalue equation*

$$\hat{\Lambda}_g[\psi_C(x,z)] = g(x)\tau_C(x)\psi_C(x,z)$$

where $\hat{\Lambda}_g \in \mathbb{T}$ is the translational-differential operator defined by

$$\hat{\Lambda}_g := z^{-n} \circ b(\bar{K}_C) \circ b(\bar{Q}_g) \circ \frac{z^n}{q_C(z)}$$

with $\bar{Q}_g$ defined as in lemma 3.1.

Proof. Formally introducing inverses [1], we determine from lemma 3.1 that

$$\pi(x) := g(x)\tau_C(x) = \bar{K}_C \circ L_0^{-1} \circ \bar{Q}$$

and hence (by applying the anti-involution b to this equation)

$$b(\pi(x)) = b(\bar{Q}) \circ \frac{1}{q_C(z)} \circ b(\bar{K}_C).$$

Then

$$\begin{aligned}\hat{\Lambda}_g[\psi_C(x,z)] &= z^{-n} \circ b(\bar{K}_C) \circ b(\bar{Q}) \circ \frac{z^n}{q_C(z)}\left[\frac{1}{z^n\tau_C(x)}\bar{K}_C e^{xz}\right] \\ &= \frac{z^{-n}}{\tau_C(x)} \circ b(\bar{K}_C) \circ b(\bar{Q}) \circ \frac{1}{q_C(z)}[\bar{K}_C e^{xz}] \\ &= \frac{z^{-n}}{\tau_C(x)} \circ b(\bar{K}_C)[\pi(x)e^{xz}] \\ &= \frac{z^{-n}\pi(x)}{\tau_C(x)} \circ \bar{K}_C[e^{xz}] \\ &= \pi(x)\psi_C(x,z).\end{aligned}$$

□

Note that according to theorem 3.1, each operator $\hat{\Lambda}_g$ satisfies an *intertwining relationship*

$$W \circ b(\pi(x)) = \hat{\Lambda}_g \circ W$$

with the constant coefficient operator $b(\pi(x))$ where $W = z^{-n} \circ b(\bar{K}_C)$. As a result we find that:

Corollary 3.1. *The set of all such operators $\hat{\Lambda}_g$ for a given subspace $C \subset \mathbb{D}$ form a commutative ring of translational-differential operators.*

4. Examples

If we choose C to be the two-dimensional space spanned by $c_1 = \Delta(1,0)$ and $c_2 = \Delta(1,1)$ (a 'two-particle' Calogero–Moser-type solution) then

$$\psi_C(x,z) = \left(1 + \frac{2 + x - (2x + x^2)z}{x^2 z^2}\right)\mathrm{e}^{xz}.$$

In this case the translational differential operators $\hat{\Lambda}$ given by theorem 3.1 are simply ordinary differential operators. For instance,

$$\hat{\Lambda} = \partial_z^3 + \frac{3}{z-z^2}\partial_z^2 - \frac{6z^2-12z+3}{z^3(z-1)^2}\partial + \frac{12z-6}{z^2(z-1)^2}$$

which satisfies $\hat{\Lambda}\psi_C(x,z) = x^3\psi_C(x,z)$ (as we would expect from earlier results on bispectrality).

However, if we had chosen instead $c_1 = \Delta(0,1)+\Delta(0,-1)$ and $c_2 = \Delta(0,2)+\Delta(0,0)$ we would have had the case of a two-soliton solution with

$$\psi_C(x,z) = \left(1 - \frac{6+(3z-2)\mathrm{e}^{2x}+2z-z\mathrm{e}^{-2x}}{(\mathrm{e}^x+\mathrm{e}^{-x})^2z^2}\right)\mathrm{e}^{xz}.$$

One finds from the procedure given in the theorem that

$$\begin{aligned}\hat{\Lambda} = z^{-2}\circ((20z+11z^2-8z^3+z^4)\boldsymbol{S}_{-3} &+ (60-68z-z^2+8z^3+z^4)\boldsymbol{S}_5\\ &+(-36+24z+16z^2-16z^3+4z^4)\boldsymbol{S}_{-1}\\ &+(-44-88z-8z^2+16z^3+4z^4)\boldsymbol{S}_3\\ &(-12-16z-2z^2+6z^4)\boldsymbol{S}_1)\circ\frac{z^2}{z^4-2z^3-z^2+2z}\end{aligned}$$

satisfies $\hat{\Lambda}[\psi_C(x,z)] = \mathrm{e}^{-3x}(1+\mathrm{e}^{2x})^4\psi_C(x,z)$.

5. Conclusions

In addition to being a generalization of the results of [5, 20] on bispectral ordinary differential operators, this paper may be seen as a generalization of [15] in which wavefunctions of n-soliton solutions of the KdV equation are shown to satisfy difference equations in the spectral parameter. The idea that KP solitons might be translationally bispectral was proposed in [14].

As in [5, 20], equations (1) and (2) lead to the well known 'ad' relations associated to bispectral pairs. That is, defining the ordinary differential operator A_m in x and the translational-differential operator $\hat{A}_m$ in z by

$$A_m = \mathrm{ad}^m_{L_p}(\pi(x)) \qquad \hat{A}_m = (-1)^m\mathrm{ad}^m_{p(z)}(\hat{\Lambda})$$

one finds that $A_m\psi_C(x,z) = \hat{A}_m\psi_C(x,z)$. Similarly, if

$$B_m = \mathrm{ad}^m_{\pi(x)}(L_p) \qquad \hat{B}_m = (-1)^m\mathrm{ad}^m_{\hat{\Lambda}}(p(z))$$

then $B_m\psi_C(x,z) = \hat{B}_m\psi_C(x,z)$. Note that whenever the order of $B_{m-1} = N > 0$ the order of B_m cannot be greater than $N-1$. So, the familiar result that $B_m \equiv 0$ and $\hat{B}_m \equiv 0$ for $m > \mathrm{ord}L_p$ holds, which is clearly a strong restriction on the operator $\hat{\Lambda}$. However, unlike the case of bispectral ordinary differential operators, one cannot conclude that $A_m \equiv 0$ for sufficiently large m since the order of $\hat{A}_m$ may not be reduced by increasing m.

The bispectrality of the rational KP solutions [20] has been shown to have a dynamical significance. In particular, it was shown that the *bispectral involution* is the linearizing map for the classical Calogero–Moser particle system [11, 20, 21]. Moreover, other bispectral KP solutions have been found to have similar properties [12, 16]. This would seem to indicate that it is likely that the bispectrality of KP solitons also has a dynamical significance, as a map between the classical Ruijsenaars and Sutherland systems (cf [17]). In fact, such a bispectral relationship between the *quantum* versions of these systems has been recently found in [6]. The dynamical significance of these results will be considered in a separate paper.

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