SYMMETRIES AND SOLUTIONS OF THE RATIONAL NESTED BETHE ANSATZ

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ABSTRACT. Recent interest in discrete, classical integrable systems has focused on their connection to quantum integrable systems via the Bethe equations. In this note, solutions to the rational nested Bethe ansatz (RNBA) equations are constructed using the "completed Calogero-Moser phase space" of matrices which satisfy a finite dimensional analogue of the canonical commutation relationship. A key feature is the fact that the RNBA equations are derived only from this commutation relationship and some elementary linear algebra. The solutions constructed in this way inherit continuous and discrete symmetries from the CM phase space.

1. INTRODUCTION

Let $\mathcal{M}_n$ be the set of all pairs of $n \times n$ complex matrices (X, Z) which satisfy the commutation relationship

$$\text{rank}([X,Z]+I)=1 \tag{1}$$

where I is the $n \times n$ identity matrix. (In other words, this says that $[Z, X] \approx I$, up to a rank one deformation.) Then $\mathcal{M}_n$ forms the "adelic Grassmannian" or "completed Calogero-Moser phase space" that arises in the study of integrable systems of mathematical physics [3, 5, 9, 11] as well as in the study of properties of rings of differential operators [1, 10]. It is the purpose of this note to establish a relationship between these matrices and the integrable system

$$\prod_{k=1}^{n} \frac{(x_j^m - x_k^{m-1})(x_j^m - x_k^m + \eta)(x_j^m - x_k^{m+1} - \eta)}{(x_j^m - x_k^{m-1} + \eta)(x_j^m - x_k^m - \eta)(x_j^m - x_k^{m+1})} = -1 \qquad \forall 1 \leq j \leq n \tag{2}$$

for functions x_j^m $(1 \leq j \leq n)$ of the discrete "time" parameter m. Note that equations (2) arise independently as the *Bethe equations* for the elementary energies of certain solvable quantum models and as a discretization of known classical particle systems [6, 7, 8].

One *philosophy* behind the recent interest in *discrete* integrable systems is that things seem to be simpler, and theorems true for more elementary reasons, in these situations. This can be seen for example from the fact that the connection between the commutation relationship (1) and the system (2) is proved by only a few elementary identities of linear algebra.

2. SOME BASIC IDENTITIES IN THE CASE $\det X = 0$

Fix a choice of $(X, Z) \in \mathcal{M}_n$ such that $\det(X) = 0$ and let $\vec{e}$ and $\vec{f}$ be vectors such that $\vec{e} \cdot \vec{f}^\top$ is the rank one matrix $[X, Z] + I$. We will use the notation $\widetilde{M}$ to denote the matrix of cofactors of the matrix M. (In particular, if M is invertible then $\widetilde{M} = \det(M)M^{-1}$.) Any matrix with determinant zero has a matrix of cofactors with rank at most one, so once again we may specify that $\vec{v}$ and $\vec{w}$ are two vectors such that $\widetilde{X} = \vec{v} \cdot \vec{w}^\top$.

It will be convenient for us to refer to the following functions

$$p(\lambda) := \vec{f}^\top \cdot (\widetilde{\lambda I - Z}) \cdot \vec{v}$$

$$q(\lambda) := \vec{w}^\top \cdot (\widetilde{\lambda I - Z}) \cdot \vec{e}$$

and the constants $\gamma := (\vec{w}^\top \cdot \vec{e})$ and $\mu := (\vec{f}^\top \cdot \vec{v})$.

Lemma 2.1. *We have:*

1. $\det[(\lambda I - Z) \cdot X + I] = \gamma p(\lambda)$,
2. $\det[X \cdot (\lambda I - Z) - I] = -\mu q(\lambda)$,
3. $\det[(\lambda_1 I - Z) \cdot X \cdot (\lambda_2 I - Z) + (\lambda_2 - \lambda_1) I] = (\lambda_2 - \lambda_1) p(\lambda_1) q(\lambda_2)$.

Proof. By the commutation relationship (1) one has that

$$(\lambda I - Z) \cdot X + I = X \cdot (\lambda I - Z) + \vec{e} \cdot \vec{f}^\top.$$

Then, it is a general fact that the determinant of a rank one perturbation of a matrix is given by the formula

$$\det(M + \vec{e} \cdot \vec{f}^\top) = \det(M) + \vec{f}^\top \cdot \widetilde{M} \cdot \vec{e}.$$

Applying this fact gives the first identity above. The others follow from the same sort of argument. □

3. Nested Bethe Ansatz Equations

For any pair $(X, Z) \in \mathcal{M}_n$ define the function

$$\tau_{X,Z}(\vec{\ell}, \vec{\lambda}) = \tau(\vec{\ell}, \vec{\lambda}) := \det(X + \sum_{i=1}^{\infty} \ell_i (\lambda_i I - Z)^{-1})$$

where $\vec{\ell} = (\ell_1, \ell_2, \dots)$ and $\vec{\lambda} = (\lambda_1, \lambda_2, \dots)$ ($\ell_i, \lambda_i \in \mathbb{C}$, only a finite number of $\ell_i \neq 0$). This is a polynomial τ-function for the KP hierarchy written in so-called *Miwa variables*. One could show that in any three of the variables ℓ_i this function satisfies the discrete Hirota equations [6]. However, here we will only observe that the linear algebra identities above show that the roots of τ satisfy the RNBA equations (2).

The RNBA equations involve only two variables: x and m. These can be taken to correspond to any two pairs (ℓ_i, λ_i). So, let us choose any two values for i: i_1 and i_2. We adopt the notation of [7], specifying a non-zero lattice spacing $\eta \in \mathbb{C}$, and writing the function τ as $\tau^m(x)$ where $x = \eta \ell_{i_1}$ and $m = \ell_{i_2} \in \mathbb{C}$.

Note that $\tau^m(x)$ is a polynomial in x (a special case of the elliptic polynomials considered in [7]). Denote by x_j^m the roots of the polynomials $\tau^m(x)$ (listed in any order). The connection between these functions and equations (2) is an immediate consequence of the following factorization of τ evaluated at specific values of its arguments:

Lemma 3.1. *Choosing either the top or bottom sign at every choice, we have the two factorization formulas:*

$$\tau^{m\mp1}(x_j^m \pm \eta) = \frac{1}{\gamma\mu} (\pm\lambda_{i_2} \mp \lambda_{i_1}) \tau^{m\mp1}(x_j^m) \tau^m(x_j^m \pm \eta).$$

Proof. Let us denote by X' the matrix

$$X' = X + \frac{x_j^m}{\eta}(\lambda_{i_1} I - Z)^{-1} + m(\lambda_{i_2} - Z)^{-1}.$$

Then, since $\tau^m(x_j^m) = 0$ we know that $\det X' = 0$. Also, it is clear that $\operatorname{rank}([X', Z] + I) = 1$. Hence we may apply the results of the lemma of the previous section to the function

$$\begin{aligned}
\tau^{m-1}(x_j^m + \eta) &= \det(X' + (\lambda_{i_1} I - Z)^{-1} - (\lambda_{i_2} I - Z)^{-1}) \\
&= \frac{1}{\det(\lambda_{i_1} I - Z)(\lambda_{i_2} I - Z)} \det((\lambda_{i_1} I - Z) \cdot X' \cdot (\lambda_{i_2} - Z) - (\lambda_{i_2} - \lambda_{i_1}) I) \\
&= \frac{\lambda_{i_2} - \lambda_{i_1}}{\det(\lambda_{i_1} I - Z)(\lambda_{i_2} I - Z)} p(\lambda_1) q(\lambda_2) \\
&= \frac{\lambda_{i_1} - \lambda_{i_2}}{\mu\gamma} \tau^m(x_j^m + \eta) \tau^{m-1}(x_j^m).
\end{aligned}$$

Note that we first used part 3 of the previous lemma to factor $\tau^{m-1}(x_j^m + \eta)$ into a constant times the polynomials $p(\lambda_1)$ and $q(\lambda_2)$ and then used parts 1 and 2 to view those as τ evaluated at other points. The other equation in which the lower sign is chosen at each place is proved similarly. □

Corollary 3.1. *The functions x_j^m satisfy the RNBA* (2) *equations.*

Proof. Using the last lemma, we can factor one term and recombine two other terms in this three term product to deduce that

$$\begin{aligned}
&\tau^{m+1}(x_j^m) \tau^m(x_j^m - \eta) \tau^{m-1}(x_j^m + \eta) \\
= \ &\tau^{m+1}(x_j^m) \tau^m(x_j^m - \eta) \left(\frac{1}{\gamma\mu} (\lambda_{i_1} - \lambda_{i_2}) \tau^{m-1}(x_j) \tau^m(x_j + \eta) \right) \\
= \ &- \left(\frac{\lambda_{i_2} - \lambda_{i_1}}{\gamma\mu} \tau^{m+1}(x_j^m) \tau^m(x_j^m - \eta) \right) \tau^{m-1}(x_j) \tau^m(x_j + \eta) \\
= \ &-\tau^{m+1}(x_j^m - \eta) \tau^{m-1}(x_j) \tau^m(x_j + \eta).
\end{aligned}$$

So, in particular,

$$\frac{\tau^{m+1}(x_j^m) \tau^m(x_j^m - \eta) \tau^{m-1}(x_j^m + \eta)}{\tau^{m+1}(x_j^m - \eta) \tau^{m-1}(x_j) \tau^m(x_j + \eta)} = -1.$$

Expanding this equation by the formula $\tau^m(x) = c \prod (x - x_j^m)$ (for some constant c) yields exactly the RNBA equations. □

As a consequence, we get our main result:

Theorem 3.1. *Let $(X, Z) \in \mathcal{M}_n$, then for any $\lambda_1, \lambda_2 \in \mathbb{C}$ (λ_2 not an eigenvalue of Z), the n eigenvalues of the matrix*

$$\mathcal{X}(m) := -\eta X \cdot (\lambda_1 - Z) - m\eta(\lambda_2 - Z)^{-1} \cdot (\lambda_1 - Z)$$

satisfy equations (2). *Note that $\mathcal{X}(m+1) - \mathcal{X}(m)$ is a constant matrix and hence we have represented solutions of* (2) *as a free flow on matrices.*

3.1. **Remarks.** As a result of this construction, several symmetries of the solution space manifest themselves. In particular, in addition to the obvious symmetries presented by translation in $\vec{\ell}$ (the "commuting flows" which appear as translations $X \to X + f(Z)$ on $\mathcal{M}_n$), there is also the "dual" flow (cf. [2, 3]) corresponding to translations of the form $Z \to Z + f(X)$. In addition, the parameters $\vec{\lambda}$ allow for continuous deformation of the solutions and the involution $(X, Z) \to (X^T, Z^T)$ on $\mathcal{M}_n$ provides discrete symmetry of the of solutions. It would be of interest to observe whether this symmetry has any special significance for this system. Recall (cf. [2, 4, 3, 9, 11]) that this same involution is a linearizing map for Calogero-Moser type particle systems and a "bispectral involution" for the Lax operators of rational solutions to the KP hierarchy [10].

As we know a great deal about the structure of $\mathcal{M}_n$ [11], this may be useful in finding new features of the Bethe ansatz equations. For instance, note that the matrix $\mathcal{X}(m)$ corresponding to the simplest 2×2 matrices satisfying (1):

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

has eigenvalues

$$\frac{-\eta\left(\lambda_1 m + \lambda_2\left(-1 + \lambda_1 + m\right)\right) \pm \sigma(\lambda_1, \lambda_2, m)}{2\lambda_2}$$

with

$$\sigma(\lambda_1, \lambda_2, m) = \sqrt{\eta^2\left(-4\lambda_1\lambda_2 m\left(-1 + \lambda_2 + m\right) + \left(\lambda_1 m + \lambda_2\left(-1 + \lambda_1 + m\right)\right)^2\right)}$$

So, in particular, in the case $\lambda_1 = \lambda_2$, they become simply

$$\{-\eta(\lambda_1 - 1 + m), -\eta m\},$$

a sort of "bound state" in which the eigenvalues remain a distance of $\eta(\lambda_1 - 1)$ units apart regardless of the value of the discrete time parameter m.

That the manifold $\mathcal{M}_n$ should be related to solutions of the RNBA equations is not a surprise. In fact, one knows from previous results that these matrices can be used to write τ-functions, that when written in Miwa form would satisfy the Hirota bilinear difference equation closely related to the RNBA equations (cf. [6] and [11]). The point of the observations above is merely that one is able to precisely determine the connection between the commutation relationship (1) and the dynamical system (2) without refering to any of these more general results. In particular, the results above provide a means for demonstrating the role of the manifold $\mathcal{M}_n$ in integrable systems using only elementary linear algebra, without any mention of symplectic geometry or Hamiltonian dynamics (as was used in [5]). It might be of interest to determine to what extent one may rederive the known results about Calogero-Moser systems in the continuum limit using only the linear algebra identities recalled here.

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