Kernel Inspired Factorizations of Partial Differential Operators

Alex Kasman
Mathematical Sciences Research Institute
1000 Centennial Drive
Berkeley, CA 94720
kasman@msri.org

Abstract: It is elementary to factor an *ordinary* differential operator given a non-zero function in its kernel. Here a *higher dimensional* generalization of this fact is developed and utilized to determine non-trivial factorizations of constant coefficient *partial* differential operators. Through Darboux transformations, this result is applied to the solution of linear partial differential equations, commutativity, quantum integrability and the "bispectral problem".

1 Introduction

The factorization of ordinary differential operators (ODOs) is quite similar to the factorization of polynomials in a single variable. If one knows a root $\lambda \in \mathbb{C}$ of some polynomial $p(x) \in \mathbb{C}[x]$ then it immediately follows that $p(x) = q(x)(x-\lambda)$ for some polynomial $q \in \mathbb{C}[x]$. Note that just as $x - \lambda$ is the only monic linear polynomial that vanishes at λ, $K = \phi(x) \circ \frac{\partial}{\partial x} \circ \phi^{-1}(x)$ is the only monic first order ODO with $\phi(x)$ in its kernel. Analogously, the knowledge that $\phi(x)$ is in the kernel of an ODO implies that K is a factor of that operator.

Theorem 1.1: *If $\phi \neq 0$ is in the kernel of the ODO $L = \sum_{i=1}^{m} f_i(x)\partial^i$ ($\partial := \frac{\partial}{\partial x}$) then it factors as $L = Q \circ K$ for some ODO Q and the first order differential operator $K = \phi(x) \circ \partial \circ \phi^{-1}(x)$.*

Moreover, since any ODO "completely factors" into first order terms, knowing the entire kernel is sufficient to determine all possible factorizations. (In general, one can say that $L = Q \circ K$ if and only if $\ker K \subset \ker L$.) This property has proved to be useful in constructing Darboux transformations for ordinary differential operators with many applications in mathematical physics *[2, 8, 9, 17, 18, 20]*.

For the same reasons, there is interest in being able to perform such Darboux transformations for partial differential operators (PDOs) *[1, 3, 5, 6, 12, 15, 22, 23]*. However, in keeping with the analogy to polynomials, the fact that one cannot necessarily completely factor a PDO complicates matters in the higher dimensional situation. Even though the analogous statement: "*If the PDO L_1 annihilates every function in the kernel of the PDO L_2 then $L_1 = Q \circ L_2$ for some PDO Q*" is true *[21]*, it does not seem to be useful in determining previously unknown factorizations.

Here we will prove a result which, although not completely general, leads to novel factorizations of constant coefficient partial differential operators. For any differentiable function $\phi(x_1,\ldots,x_n)$ and irreducible polynomial $q(z_1,\ldots,z_n)$ we will introduce below a partial differential operator K_q^ϕ and a proper subset $F_q^\phi \subset \ker K_q^\phi$. The main theoretical result of this paper is to prove that *any* PDO whose kernel contains F_q^ϕ has K_q^ϕ as a right factor. Using this fact, which has the well known result Theorem 1.1 as a corollary, we are able to determine new, explicit, non-trivial factorizations of constant coefficient partial differential operators. The application of such factorizations to the solution of linear PDEs, bispectrality and quantum integrability through Darboux transformations is discussed.

1.1 NOTATION

Let $\mathcal{D}$ be the ring of partial differential operators in the variables $\vec{x}=(x_1,\ldots,x_n)$ and coefficients in some differential field $\mathbf{K}$ of functions. The shorthand notations $\partial_i := \frac{\partial}{\partial x_i}$ and $\vec{\partial}=(\partial_1,\ldots,\partial_n)$ will be used. (So $\mathcal{D}=\mathbf{K}[\vec{\partial}]$ is made up of polynomials in the ∂_i's with coefficients that are functions in $\mathbf{K}$.) Differential operators act on functions by differentiation and multiplication in the obvious way, and the action of the operator $L\in\mathcal{D}$ on a function $f(x_1,\ldots,x_n)$ will be denoted $L[f]$. Moreover, the composition of operators $L_1,L_2\in\mathcal{D}$ will be written $L_1\circ L_2$.

We will say that a function $f(x_1,\ldots,x_n)$ is *polynomial-exponential* if it is of the form

$$f(x_1,\ldots,x_n)=\sum_{i=1}^{N} p_i(x_1,\ldots,x_n)e^{\langle\vec{x},\vec{\alpha}_i\rangle} \tag{1.2}$$

for some $N\in\mathbb{N}$, polynomials $p_i\in\mathbb{C}[x_1,\ldots,x_n]$ and $\vec{\alpha}_i\in\mathbb{C}^n$. (Here $\langle\,,\,\rangle$ denotes the usual inner product on $\mathbb{C}^n$.) Note that the set of polynomial-exponential functions forms a ring which is moreover closed under differentiation. Similarly, the quotient field of ratios of such functions with non-zero denominators is also closed under differentiation. Let us call this quotient field the *rational-exponential* functions. This paper is primarily concerned with the ring of partial differential operators in the variables x_i with coefficients that are rational-exponential functions as defined above. More specifically, the main result involves the factorization of constant coefficient differential operators in this ring.

In addition, in order to prove the results below it will be useful to introduce the ring $\mathbb{T}$ of rational coefficient translational-differential operators in $\vec{z}=(z_1,\ldots,z_n)$ (cf. *[19]* which considers the one dimensional case). An element $\Lambda\in\mathbb{T}$ is an operator of the form

$$\Lambda=\sum_{i=1}^{N} L_i\circ\mathbf{T}_{\vec{\alpha}_i} \tag{1.3}$$

where $\mathbf{T}_{\vec{\alpha}}$ is the translation operator

$$\mathbf{T}_{\vec{\alpha}}[f(\vec{z})]=f(\vec{z}+\vec{\alpha})\qquad \vec{\alpha}\in\mathbb{C}^n \tag{1.4}$$

and L_i are rational coefficient differential operators in the variables z_i

$$L_i = \sum_{j=1}^{m} r_{ij}(\vec{z}) \circ p_{ij}(\hat{\partial}_1, \ldots, \hat{\partial}_n), \ \hat{\partial}_i = \frac{\partial}{\partial z_i}, \ r_{ij} \in \mathbb{C}(\vec{z}), \ p_{ij} \in \mathbb{C}[\vec{z}]. \tag{1.5}$$

Note that $\mathbb{C}[\vec{z}] \subset \mathbb{T}$ by associating $p \circ \mathbf{T}_{\vec{0}} \in \mathbb{T}$ to the polynomial p. Special attention should be paid to the subring $\mathbb{T}^0$ of *constant* coefficient translational-differential operators (i.e. those for which $L_i \in \mathbb{C}[\hat{\partial}_1, \ldots, \hat{\partial}_n]$). The main significance of these operators here comes from the following observation:

Lemma 1.6: *The map I from the ring $\mathbb{T}^0$ of constant coefficient translational-differential operators in $\vec{z}$ to the ring of polynomial-exponential functions in $\vec{x}$ given by*

$$\Lambda[e^{\langle \vec{x}, \vec{z} \rangle}] = I(\Lambda) e^{\langle \vec{x}, \vec{z} \rangle} \qquad \Lambda \in \mathbb{T}^0 \tag{1.7}$$

is an isomorphism.

In particular, $e^{\langle \vec{x}, \vec{z} \rangle}$ is an eigenfunction for every element of $\mathbb{T}^0$ and the eigenvalues are polynomial-exponential functions.

2 Factoring Partial Differential Operators

Let $\phi(x_1, \ldots, x_n) \in \mathbf{K}$ be an arbitrary non-zero differentiable function and $q(z_1, \ldots, z_n)$ be an *irreducible* polynomial. We will associate to this choice a differential operator and a set of functions.

Definition 2.1: Let $K_q^\phi \in \mathcal{D}$ be the differential operator

$$K_q^\phi := \phi(\vec{x}) \circ q(\vec{\partial}) \circ \frac{1}{\phi(\vec{x})}.$$

Note in particular that this operator annihilates every function in the set of functions

$$F_q^\phi := \left\{ \phi(\vec{x}) e^{\langle \vec{x}, \vec{z} \rangle} \mid \vec{z} \in \mathbb{C}^n, \ q(\vec{z}) = 0 \right\}$$

parametrized by the irreducible algebraic variety $q^{-1}(0)$.

Theorem 2.2: *$L \in \mathcal{D}$ factors as*

$$L = Q \circ K_q^\phi \qquad \textit{for some } Q \in \mathcal{D}$$

if and only if $F_q^\phi \subset \ker L$.

Proof: Clearly if L has a right factor of K_q^ϕ then $F_q^\phi \subset \ker L$. So, let us assume the latter and prove the former.

Consider first the case that $\phi(\vec{x}) \equiv 1$ and so $K_q^\phi = q(\vec{\partial})$. Note that

$$f(\vec{x}, \vec{z}) := L[e^{\langle \vec{x}, \vec{z} \rangle}] = \sum f_i(\vec{x}) p_i(\vec{z}) e^{\langle \vec{x}, \vec{z} \rangle}$$

for some polynomials p_i and functions f_i. Then, for any fixed value of $\vec{x}$ in its domain, this function must vanish for all $\vec{z} \in q^{-1}(0)$. Consequently, by the Nullstellensatz, the polynomial (in $\vec{z}$) $f(\vec{x},\vec{z})e^{\langle -\vec{x},\vec{z}\rangle}$ has a factor of $q(\vec{z})$ and so we may factor

$$f(\vec{x},\vec{z}) = q(\vec{z}) \sum \hat{f}_i(\vec{x})\hat{q}_i(\vec{z})e^{\langle \vec{x},\vec{z}\rangle}. \tag{2.3}$$

Then, L factors as $L = Q \circ q(\vec{\partial})$ with

$$Q = \sum \hat{f}_i(\vec{x})\hat{q}_i(\vec{\partial}). \tag{2.4}$$

More generally, for arbitrary ϕ let $\hat{L} = L \circ \phi \in \mathcal{D}$, then by the preceding paragraph we have that $\hat{L} = \hat{Q} \circ q(\vec{\partial})$. Hence $L = \hat{L} \circ \frac{1}{\phi} = \hat{Q} \circ q(\vec{\partial}) \circ \frac{1}{\phi}$. Finally, letting $Q = \hat{Q} \circ \frac{1}{\phi}$ we get the desired factorization. □

Remark 1: Note that if $n = 1$ and $q(z) = z \in \mathbb{C}[z]$ then $F_z^{\phi(x)} = \{\phi(x)\}$ and $K_z^{\phi(x)} = \phi \circ \partial \circ \phi^{-1}$. In particular, in this case one finds Theorem 1.1 as a *consequence* of Theorem 2.2.

Remark 2: Theorem 2.2, though quite simple once it has been read and understood, is particularly surprising since it does not require that L annihilate *all* of $\ker K_q^\phi$ but only the (proper) subspace spanned by the elements of F_q^ϕ. This is a *stronger* statement than one might expect. (For instance note that if $\phi \equiv 1$ and $q(x_1,x_2) = x_1^2 - x_2$ then $x_1 \in \ker K_q^\phi$ but cannot be written as a linear combination of the functions in $F_q^\phi \subset \ker K_q^\phi$.)

Remark 3: Of course, most differential operators will not have any set of the form F_q^ϕ in their kernels, and so this theorem is not entirely general. However, it has the advantage of being useful. The special instance of this theorem in the case that q is linear was used to construct Darboux transformations in *[5]* and the case in which $\phi(\vec{x}) \in \mathbb{C}[\vec{x}]$ is a polynomial is implicitly contained in *[3]*. It will be further applied in the next section to determine new factorizations of constant coefficient differential operators.

3 Constant Coefficient Differential Operators

3.1 One Dimensional Case

In the case of constant coefficient *ordinary* differential operators, Theorem 1.1 implies the following:

Theorem 3.1: *There exists a constant coefficient ODO with right factor of* $K = \phi(x) \circ \partial \circ \phi^{-1}(x)$ *iff* $\phi(x) = \sum_{i=1}^n p_i(x)e^{\lambda_i x}$ *for some* $p_i \in \mathbb{C}[x]$ *and* $\lambda_i \in \mathbb{C}$.

Proof: Let ϕ be of this form and $p(x) = \prod (x-\lambda_i)^{1+\deg p_i} \in \mathbb{C}[x]$, then the operator $L = p(\partial)$ has $\phi(x)$ in its kernel. By the discussion above, this implies that L has a right factor of the form K.

Alternatively, given a constant coefficient operator $L = p(\partial)$ ($p \in \mathbb{C}[x]$) let λ_i ($1 \le i \le n$) denote the distinct roots of p and m_i be their multiplicities. Then it is simple to check that $x^j e^{\lambda_i x} \in \ker L$ for $0 \leq j \leq m_i - 1$ and $1 \leq i \leq n$. Over $\mathbb{C}$, these functions span a (deg p)-dimensional space which must therefore be all of $\ker L$. Consequently, every function in the kernel of a constant coefficient ODO is of the given form. $\square$

It will be seen that the situation is much the same in n dimensions if one replaces ϕ with any similar function of n variables and replaces ∂ with any irreducible constant coefficient partial differential operator.

3.2 Higher Dimensional Case

If one were to pick an arbitrary partial differential operator K, it would be unusual to be able to find another operator Q such that $Q \circ K$ is a constant coefficient operator. The following theorem, analogous to Theorem 3.1, provides a class of non-constant coefficient operators which are right factors of constant coefficient differential operators.

Theorem 3.2: *Let $\phi(\vec{x}) \neq 0$ be a polynomial-exponential function and $q(\vec{x}) \in \mathbb{C}[\vec{x}]$ be an irreducible polynomial, then there exists a partial differential operator $Q \neq 0$ such that $Q \circ K_q^\phi \in \mathbb{C}[\vec{\partial}]$ is a constant coefficient operator.*

Proof: Let

$$\phi(\vec{x}) = \sum_{i=1}^{m} p_i(\vec{x}) e^{\langle \vec{x}, \vec{\alpha}_i \rangle} \tag{3.3}$$

for arbitrary polynomials p_i and vectors $\vec{\alpha}_i \in \mathbb{C}^n$. Then the translational-differential operator $T = I^{-1}(\phi) \in \mathbb{T}$ satisfying $Te^{\langle \vec{x}, \vec{z} \rangle} = \phi(\vec{x}) e^{\langle \vec{x}, \vec{z} \rangle}$ (cf. Lemma 1.6) is given by

$$T = \sum_{i=1}^{m} p_i(\vec{\partial}_z) \mathbf{T}_{\vec{\alpha}_i}. \tag{3.4}$$

Now let $p(z) \in \mathbb{C}[z]$ be the polynomial

$$p(z) := \prod_{i=1}^{m} \mathbf{T}_{-\vec{\alpha}_i} \left[(q(\vec{z}))^{m_i+1} \right] \qquad m_i := \text{deg } p_i \tag{3.5}$$

and L be the constant coefficient differential operator $L = p(\vec{\partial}) \in \mathbb{C}[\vec{\partial}]$. Then

$$L[\phi e^{\langle \vec{x}, \vec{z} \rangle}] = T \circ L[e^{\langle \vec{x}, \vec{z} \rangle}] = T \circ p(z)[e^{\langle \vec{x}, \vec{z} \rangle}] \tag{3.6}$$

and $p(z)$ is constructed specifically so that $T \circ p(z) = q(z) \circ T'$ for some other translational-differential operator T' with *polynomial* coefficients. Consequently, $L[\phi e^{\langle \vec{x}, \vec{z} \rangle}] \equiv 0$ for all $\vec{z} \in q^{-1}(0)$ and then Theorem 2.2 implies that L has a right factor of K_q^ϕ. $\square$

4 Examples, Applications and Comments

Without knowledge of its kernel, factorization of even an ordinary differential operator can be difficult (cf. *[13, 14]*). If one does know functions in the kernel, the problem of factorization becomes trivial but such factorizations have application in the construction of *Darboux transformations [2, 8, 9, 16, 17, 18, 20]*, which is essentially conjugation in the ring of differential operators. Despite the inherent difficulty in factoring PDOs there has recently been much interest in the higher dimensional generalizations of the Darboux transformation *[1, 3, 12, 15, 22, 23]*. By using the factorizations determined above to perform Darboux transformations for constant coefficient PDOs one may construct new explicit higher dimensional examples of commutativity, quantum integrability and bispectrality.

Pick an irreducible polynomial $q \in \mathbb{C}[\vec{z}]$, polynomials $p_i \in \mathbb{C}[\vec{x}]$ and vectors $\vec{\alpha}_i$ for $1 \leq i \leq m \in \mathbb{N}$. Then defining ϕ by (3.3) and p by (3.5), it follows that the constant coefficient differential operator $L = p(\vec{\partial}) \in \mathbb{C}[\vec{\partial}]$ factors as $L = Q \circ K_q^\phi$ for some Q. Knowing this, one may then compute the coefficients of Q explicitly (for instance, with a computer algebra package) since L and K_q^ϕ are known.

Example: For $n = 2$, consider the case that $q(\vec{z}) = z_1^2 - z_2$, $p_1 = 1, p_2 = \gamma \in \mathbb{C}[\vec{x}]$, $\vec{\alpha}_1 = (1, 0)$ and $\vec{\alpha}_2 = (0, 1)$. Using (3.3) and (3.5) one finds that

$$L = ((\partial_1 - 1)^2 - \partial_2)(\partial_1^2 - \partial_2 + 1)$$

$$\phi(\vec{x}) = e^{x_1} + \gamma e^{x_2}$$

and

$$K_q^\phi = \partial_1^2 - \partial_2 - \frac{2e^{x_1}}{\phi}\partial_1 + 1 - \frac{2\gamma e^{x_1+x_2}}{\phi^2}.$$

As predicted, one may check that $L[\phi e^{\langle \vec{x}, \vec{z}\rangle}]$ vanishes for any $\vec{z} \in q^{-1}(0)$ and so by Theorem 2.2 $L = Q \circ K_q^\phi$. It is then merely an elementary calculation to check that

$$Q = \partial_1^2 - \partial_2 - \frac{2\gamma e^{x_2}}{\phi}\partial_1 + \frac{2\gamma e^{x_1+x_2}}{\phi^2} + 1.$$

Darboux Transformations for Linear PDEs: Given the factorization $L = Q \circ K_q^\phi$ determined above, we have a *Darboux* transformation relating the solutions to the equation $L[f] = \lambda f$ ($\lambda \in \mathbb{C}$) and the new equation $\tilde{L}[\tilde{f}] = \lambda \tilde{f}$ for $\tilde{L} = K_q^\phi \circ Q$. In particular, it is clear that the application of the operators K_q^ϕ and Q provides maps between $\ker L - \lambda$ and $\ker \tilde{L} - \lambda$.

Theorem 4.1: *If $L = Q \circ K_q^\phi$ and $\tilde{L} = K_q^\phi \circ Q$ then we have the maps*

$$K_q^\phi : \ker L - \lambda \to \ker \tilde{L} - \lambda$$
$$f \mapsto K_q^\phi[f]$$

$$Q : \ker \tilde{L} - \lambda \to \ker L - \lambda$$
$$\tilde{f} \mapsto Q[\tilde{f}]$$

between the kernels of the operators.

Proof: Suppose $f \in \ker L - \lambda$ then

$$\begin{aligned}(\tilde{L}-\lambda)[\tilde{f}] &= K_q^{\phi} \circ Q \circ K_q^{\phi}[f] - \lambda K_q^{\phi}[f] \\ &= K_q^{\phi} \circ L[f] - \lambda K_q^{\phi}[f] \\ &= K_q^{\phi}[\lambda f] - \lambda K_q^{\phi}[f] = 0\end{aligned}$$

(and similarly for the map in the other direction). □

As in other applications of Darboux transformations to differential operators *[1, 3, 8, 9, 15, 20, 23]*, these are most useful if they can be shown to preserve some form or property of interest. Here, for instance, the bispectral property and quantum integrability are considered although it seems likely that the transformations described here could be used as well for other purposes. Unfortunately, since the procedure described above depends explicitly on the fact that the initial operator has constant coefficients, it is not clear how one can *iterate* these transformations except in trivial circumstances.

Example (cont'd): Continuing the example begun above, we have the new differential operator

$$\begin{aligned}\tilde{L} = {\partial_2}^2 - 2{\partial_1}^2\partial_2 + 2\partial_1\partial_2 - 2\partial_2 + {\partial_1}^4 - 2{\partial_1}^3 + \frac{2\left(e^{2x_1} + 6\gamma e^{x_1+x_2} + e^{2x_2}\gamma^2\right)}{\left(e^{x_1} + e^{x_2}\gamma\right)^2}{\partial_1}^2 \\ + \frac{-2\left(e^{3x_1} + 9\,e^{2x_1+x_2}\,\gamma - 3e^{x_1+2x_2}\,\gamma^2 + e^{3x_2}\,\gamma^3\right)}{\left(e^{x_1} + e^{x_2}\,\gamma\right)^3}\partial_1 \\ + \frac{e^{4x_1} + 8\gamma e^{3x_1+x_2} - 10e^{2(x_1+x_2)}\gamma^2 + 8e^{x_1+3x_2}\gamma^3 + e^{4x_2}\gamma^4}{\left(e^{x_1} + e^{x_2}\gamma\right)^4}\end{aligned}$$

and the relationship Theorem 4.1 relating the solutions of the corresponding differential equations. For instance, it is trivial to find a solution to $L[f] = 0$ by assuming $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$. One such function is $f = \kappa_1 x_1 e^{x_1} + \kappa_2 x_2 e^{x_2} \in \ker L$. However, the transformation

$$\tilde{f} := K_q^{\phi}[f] = (2\kappa_1\gamma - \kappa_2)e^{x_2} - \frac{2\gamma^2 e^{3x_2}(\kappa_1\gamma x_1 - \kappa_2 x_2)}{\phi^2} + \frac{2\gamma e^{2x_2}(\kappa_1\gamma(x_1 - 1) - \kappa_2 x_2)}{\phi}$$

provides a solution to $\tilde{L}[\tilde{f}] = 0$ which would be difficult to find using other methods.

Commutativity and Quantum Integrability: A factorization of one constant coefficient operator L provides a means for producing a set of commuting operators. Let $M_i \in \mathbb{C}[\vec{\partial}]$ be constant coefficient partial differential operators. Then the (generically non-constant) operators $\tilde{L}_i := K_q^{\phi} \circ M_i \circ Q$ mutually commute. Explicitly, one sees that

$$\begin{aligned}[\tilde{L}_i, \tilde{L}_j] &= K_q^{\phi} \circ M_i \circ Q \circ K_q^{\phi} \circ M_j \circ Q - K_q^{\phi} \circ M_j \circ Q \circ K_q^{\phi} \circ M_i \circ Q \\ &= K_q^{\phi} \circ M_i \circ L \circ M_j \circ Q - K_q^{\phi} \circ M_j \circ L \circ M_i \circ Q = 0\end{aligned}$$

by the commutativity of L and $M_i \in \mathbb{C}[\vec{\partial}]$. Consequently, letting $M_0 = 1$ and $M_i = \partial_i$ ($1 \leq i \leq n$) the ring generated by $\tilde{L}_0, \cdots, \tilde{L}_n$ is a commutative ring with

Krull dimension n which cannot be generated by fewer than $n+1$ generators. Viewing the commutativity of these operators as the quantum analogue of the Poisson commutativity of functions on the cotangent bundle to a symplectic manifold, such a ring may be considered as an example of a quantum integrable system *[4, 5, 6, 12]*.

Bispectrality: The function $e^{\langle\vec{x},\vec{z}\rangle}$ is, of course, an eigenfunction for the ring of constant coefficient partial differential operators in $\vec{x}$ with eigenvalues depending on the parameters $\vec{z}$. Moreover, this same function is also an eigenfunction for the constant coefficient translational-differential operators $\mathbb{T}^0$ in $\vec{z}$ with eigenvalues depending upon $\vec{x}$. This is an instance of "bispectrality" *[7, 10, 11, 19, 22]*. Then, it is interesting to note that the function $K_q^\phi[\exp\langle\vec{x},\vec{z}\rangle]$ is *also* a common eigenfunction for differential operators in $\vec{x}$ and translational-differential operators in $\vec{z}$. The one dimensional case ($n=1$) is particularly interesting since it indicates a sort of bispectrality for solitons of the KP hierarchy and will be considered separately in *[19]*. (See also *[7]* which considers such translational-differential bispectrality in the context of quantum duality.)

Theorem 4.2: *The function $\psi := K_q^\phi[\exp\langle\vec{x},\vec{z}\rangle]$ satisfies the eigenvalue equations*

$$\tilde{L}[\psi(\vec{x},\vec{z})] = p(\vec{z})\psi(\vec{x},\vec{z}) \qquad \Lambda[\psi(\vec{x},\vec{z})] = \phi^m(\vec{x})\psi(\vec{x},\vec{z}) \tag{4.3}$$

for $\tilde{L} = K_q^\phi \circ Q \in \mathcal{D}$, $m =$ deg $q+1$ and some $\Lambda \in \mathbb{T}$.

Proof: The key observation here is that

$$\begin{aligned}\psi(\vec{x},\vec{z}) &= \phi(\vec{x}) \circ q(\vec{\partial}) \circ \frac{1}{\phi(\vec{x})}[e^{\langle\vec{x},\vec{z}\rangle}] \\ &= \frac{\hat{\phi}(\vec{x})}{\phi^{\text{deg }q}(\vec{x})} e^{\langle\vec{x},\vec{z}\rangle}\end{aligned}$$

where $\hat{\phi}(\vec{x})$ is another polynomial-exponential function of the form

$$\hat{\phi}(\vec{x}) = \sum_{i=1}^{\hat{m}} \hat{p}_i(\vec{x}) e^{\langle\vec{x},\hat{\alpha}_i\rangle}.$$

Then, letting

$$\Lambda = \left(\sum_{i=1}^{\hat{m}} \hat{p}_i(\hat{\partial}_1,\ldots,\hat{\partial}_n) \circ \mathbf{T}_{\hat{\alpha}_i}\right) \circ \frac{1}{q(\vec{z})} \circ \left(\sum_{i=1}^{m} p_i(\hat{\partial}_1,\ldots,\hat{\partial}_n) \circ \mathbf{T}_{\vec{\alpha}_i}\right)$$

it follows that Λ satisfies (4.3). $\square$

Example (cont'd): Returning to the example above, one may check that $\psi(\vec{x},\vec{z}) := K_q^\phi[\exp\langle\vec{x},\vec{z}\rangle]$ is also given by $\psi = \frac{1}{\phi^2}\Gamma[e^{\langle\vec{x},\vec{z}\rangle}]$ where $\Gamma \in \mathbb{T}$ is the translational-differential operator

$$\Gamma = \gamma^2(1+z_1^2-z_2)\mathbf{T}_{(0,2)} + (1-2z_1+z_1^2-z_2)\mathbf{T}_{(2,0)} + 2\gamma(z_1^2-z_1-z_2)\mathbf{T}_{(1,1)}.$$

Then if $\Lambda = \Gamma \circ \frac{1}{q(\vec{z})} \circ T$ with $T = I^{-1}(\phi) = \mathbf{T}_{(1,0)} + \gamma \mathbf{T}_{(0,1)}$ one finds that $\Lambda[\psi(\vec{x}, \vec{z})] = \phi(\vec{x})^3 \psi(\vec{x}, \vec{z})$.

Since this bispectrality is not only a *consequence* of the transformation but also used in the construction itself, this result is similar to *[12]* where bispectrality of partial differential operators is utilized to perform Darboux transformations. In particular, this paper extends that same idea to the case of translational bispectrality where the proof presented in *[12]* fails.

Acknowledgements: I am indebted to M. Singer and S. Tsarev for helpful discussions and to Yu. Berest and E. Horozov for continuing assistance and prior collaboration *[3, 12]*.

References

[1] A. Andrianov, N. Borisov and M. Ioffe, "The factorization method and quantum systems with equivalent energy spectra", *Physics Letters A*, 105 (1984) no 1,2 pp. 19–22.

[2] B. Bakalov, E. Horozov, M. Yakimov, "Bäcklund-Darboux transformations in Sato's grassmannian, *Serdica Math. Jour.* 22 (1996) 571-588.

[3] Yu. Berest and A. Kasman "D-modules and Darboux Transformations", *Lett. Math. Phys.* 43 (1998) 279–294

[4] A. Braverman, P. Etingof and D. Gaitsgory, "Quantum Integrable Systems and Differential Galois Theory", *Transformation Groups* 2 (1997) pp. 31–56

[5] O.A. Chalykh and A.P. Veselov, "Commutative rings of partial differential operators and Lie algebras", *Comm. Math. Phys.* 125 (1990) 597-611.

[6] O.A. Chalykh and A.P. Veselov, "Integrability in the theory of the Schrödinger operators and harmonic analysis", Comm. Math. Phys. 152 (1993) pp. 29-40

[7] O.A. Chalykh, "Duality of the generalized Calogero and Ruijsenaars problems", *Uspekhi Mat. Nauk* 52 (1997), no. 6(318) pp. 191–192

[8] G. Darboux , "Sur une proposition relative aux equation lineaires" *Compt. Rend.* **94** (1882), 1456

[9] G. Darboux, "Leçons sur la théorie générale des surfaces." 2éme partie, Paris, Gauthiers-Villars, 1889.

[10] J.J. Duistermaat and F.A. Grünbaum, "Differential equations in the spectral parameter", *Comm. Math. Phys.* 103 (1986) 177-240

[11] F.A. Grünbaum, "Bispectral Musings" CRM Proceedings and Lecture Notes, volume 14, American Mathematical Society, Providence, RI, 1988 pp. 31–46.

[12] E. Horozov and A. Kasman, "Darboux Transformations for Bispectral Quantum Integrable Systems", submitted for publication. Available as math-ph/9806002.

[13] M. van Hoeij, "Formal solutions and factorization of differential operators with power series coefficients" *J. Symbolic Comput.* 24 (1997), no. 1, pp. 1–30.

[14] M. van Hoeij, "Factorization of differential operators with rational functions coefficients" *J. Symbolic Comput.* 24 (1997), no. 5, pp. 537–561.

[15] A. Gonzalez-Lopez and N. Kamran, "The Multidimensional Darboux Transformation", `hep-th/961200`.

[16] A. Kasman and M. Rothstein, "Bispectral darboux transformations: the generalized Airy case", *Physica D* 102 (1997) p. 159-173.

[17] A. Kasman, "Darboux transformations from n-KdV to KP", *Acta Applicandae Mathematicae* 49 no. 2 (1997) pp. 179–197.

[18] A. Kasman, "Darboux Transformations and the Bispectral Problem", CRM Proceedings and Lecture Notes, volume 14, American Mathematical Society, Providence, RI, 1988 pp. 81–92.

[19] A. Kasman, "Spectral Difference Equations Satisfied by KP Soliton Wavefunctions", *Inverse Problems* 14 (1998) 1481-1487.

[20] V.B. Matveev and M.A. Salle, Darboux Transformations and Solitons, Springer Ser. Nonlinear Dynam., Springer-Verlag, Berlin, 1991.

[21] M. Singer, private communication.

[22] A. P. Veselov, Baker-Akhiezer functions and the bispectral problem in many dimensions, CRM Proceedings and Lecture Notes, volume 14, American Mathematical Society, Providence, RI, 1998, pp. 123–129.

[23] B.N. Zakhariev and V.M. Chabanov "New situation in quantum mechanics (wonderful potentials from the inverse problem)", *Inverse Problems* 13 (1997) R47–R79.