Duality and Collisions of Harmonically Constrained Calogero Particles

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Abstract. The Calogero model in a harmonic potential is known to be an integrable particle system exhibiting an unusual form of duality. Following a brief overview of the previous work relating the action-angle duality of particle systems to bispectrality, this paper applies the rank one matrix method that has proved so successful in that case to this other form of duality. In particular, the particles are found as eigenvalue dynamics for flows on a pair of manifolds of matrix pairs satisfying a rank one condition (one manifold for each possible sign of the coupling constant). Since the Jordan form of the matrices is not restricted, this allows the continuation of the dynamics through collisions where the Hamiltonian is undefined. Moreover, several simple algebraic maps between these manifolds are shown to have dynamical significance, including one which corresponds to the duality.

1. Introduction

1.1. General Remarks on Duality and Bispectrality. *(This section represents a brief summary of the contents of my talk from the session at the 2012 Joint Mathematics Meeting in Boston for which this volume records the proceedings. It also serves as motivation for the new material discussed beginning in Section 1.2.)*

Duality is an interesting phenomenon wherein two seemingly different questions turn out to be mathematically equivalent. A simple example is the equivalence between selecting k or $n-k$ objects from a set of n objects. A similar but conceptually more difficult one is the equivalence in mathematical physics between k electrons in a "Fermi sea" and the situation in which all but k possible energy states are filled by positrons. (This is sometimes referred to as a "particle-hole duality". See, for example, [**17**].) Such dualities can be useful when they help us replace a difficult computation with an equivalent simpler one, but also lead to the philosophical question of whether they indicate a fundamental equivalence between the two situations at an existential level as well. (E.g. is there really a difference between the presence of an particle and the absence of its anti-particle, or is it just a matter of interpretation?)

A standard duality in classical integrable particle systems is the so-called "action-angle duality". An integrable particle system can be identified by its action-angle map between the symplectic manifold in which it "lives" and another in which its dynamics simplifies. However, pushing simple dynamics on the first manifold forward through the action-angle map produces dynamics in the image which are simplified by the inverse of the map. In this way, integrable systems have a natural duality pairing systems whose action-angle maps are inverses. For example, the (rational) Calogero-Moser system has an action-angle map which is an involution [**2**] and hence that system is self-dual, but a less trivial example is the duality between the Ruijsenaars-Schneider and the hyperbolic Calogero-Moser systems [**19**].

A major theme in my research has been the connection between the action-angle duality of integrable particle systems and *bispectrality*. A linear operator L acting on functions of the parameter(s) x is said to be "bispectral" if one can find an eigenfunction $f(x,z)$ satisfying $Lf = p(z)f$ with an eigenvalue depending on the spectral parameter(s) z and also an operator Λ acting on functions of z such that $\Lambda f = \pi(x)f$ [**9**]. (Here it is assumed that L is independent of z, Λ is independent of x and both p and π are non-constant functions.) Although there is no obvious dynamic content to the definition, bispectrality has been linked to integrable systems since it was first introduced by Duistermaat and Grünbaum in a paper that connected it to the KdV equation [**5**].

When a classical particle system is quantized, its Hamiltonian becomes an operator. Ruijsenaars observed that letting L and Λ be the Hamiltonians of a classical particle system and its dual partner respectively produced operators satisfying the eigenvalue equations of bispectrality with a common eigenfunction [**19**]. This has essentially become the definition of duality for quantum systems [**6, 10, 14**].

More mysteriously, *classical* duality also seems to be manifested in the form of bispectrality in an entirely different way: the particle systems are associated to the dynamics of a soliton equation in the special case that the Lax operator is bispectral, and the exchange of the Lax operator with its bispectral partner is the action-angle map between the particle systems. This was first observed for the (rational) Calogero-Moser system [**11, 20**]. With only that one example, it certainly was reasonable to think that the relationship between action-angle duality and bispectrality in this case was merely a coincidence. However, I have been pursuing this as a "program" since that time [**12**] and have collected enough examples to hopefully convince others that there is something interesting going on here that is not yet fully understood.

My talk in this session was an overview of the present status of this research program with special emphasis on two recent developments each of which extended the duality/bispectrality correspondence to an area in which there was previously reason to doubt it would work.

Until recently, all examples of duality of classical particle systems being manifested as bispectrality for Lax operators involved *self-dual* systems. With the intention of using these constructions as a basis for extending the correspondence to the (non-self-dual) case of the duality between the Ruijsenaars-Schneider and hyperbolic Calogero-Moser systems I first demonstrated that KP solitons are bispectral [**13**] and with Michael Gekhtman applied the rank one matrix formulation to KP solitons [**16**]. Unfortunately, I was unable to see a way to connect them to the action-angle map relating these two particle systems and began to doubt that it could be done. I am therefore pleased to report that Luc Haine [**8**] did indeed show that the action-angle duality of these systems is manifested in the bispectrality of KP solitons under translational operators in the spectral parameter as in [**13**] and is nicely described using "almost-intertwining" matrices as in [**16**].

Another new development concerns the connection between bispectrality for *matrix* differential operators and the self-duality of the spin generalization of the Calogero-Moser system. Until recently it seemed unlikely that the correspondence would continue to be found in this case, since previous investigation had found little to say about bispectrality for such matrix operators. Merely replacing the scalar coefficients with matrix coefficients on differential operators resulted in a comparatively small and uninteresting set of solutions to the bispectral problem. However, Maarten Bergvelt, Michael Gekhtman and I fully generalized the bispectrality/duality correspondence from the scalar case to the spin particle system [**3**]. (In fact, our result here was just a tiny bit stronger in regard to particle collisions and so even produced new results in the scalar case.) A simple modification to the bispectral problem under consideration turned out to be an essential step: it is necessary to view the matrix operators in the spectral parameter as acting from the *right* rather than the left in order to preserve the richness of the bispectral problem.

That the action-angle duality of integrable systems can be manifested in the form of bispectral Lax operators for soliton equations even in the non-self-dual and matrix cases lends further weight to the two main unanswered questions in this research program: Can we prove a theorem that would demonstrate the general equivalence of the bispectral involution (exchanging L and Λ) and action-angle maps for classical integrable systems? Is it a coincidence that duality is manifested as bispectrality at *both* the quantum and classical levels?

For more details regarding my talk on the correspondence between bispectrality and action-angle duality of particle systems, I recommend that the interested reader obtain a copy of my talk, which is presently available on my Website [**15**]. However, at this point the topic of the paper will change to a *different* sort of duality.

1.2. Calogero Systems Constrained by a Harmonic Potential. Consider the classical dynamical system with Hamiltonian

$$\mathcal{H}_{hCM} = \frac{1}{2}\sum_{j=1}^{n}(\dot{x}_j^2 + \omega^2 x_j^2) - \sum_{j\neq k}\frac{\gamma^2}{(x_j - x_k)^2} \tag{1.1}$$

where, of course, x_j are the positions of the n particles of unit mass and $\dot{x}_j = \frac{d}{dt}x_j$ are their momenta. The number γ is the "coupling constant" and ω determines the strength of an external harmonic potential. If all of the parameters take real values and ω is non-zero then the dynamics are completely periodic and bounded [**4, 21**]. (In the case $\omega = 0$, this is the (rational) Calogero-Moser particle system whose self-duality is manifested in the bispectrality of rational KP solutions [**11, 20**] as described above.)

A different sort of duality for the harmonic case is considered by Abanov, Gromov and Kulkarni [**1**]. In particular, they show that if $\{x_1, \ldots, x_n\}$ are governed by this Hamiltonian then there exists another collection of functions $\{z_1, \ldots, z_m\}$ which in *in addition* to having dynamics governed by the same Hamiltonian, also satisfies the coupled equations

$$\dot{x}_j - i\omega x_j = -i\gamma \sum_{k\neq j} \frac{1}{x_j - x_k} + i\gamma \sum_{k=1}^{n} \frac{1}{x_j - z_k} \qquad 1 \leq j \leq n \tag{1.2}$$

$$\dot{z}_j - i\omega z_j = i\gamma \sum_{k\neq j} \frac{1}{z_j - z_k} - i\gamma \sum_{k=1}^{m} \frac{1}{z_j - x_k} \qquad 1 \leq j \leq m. \tag{1.3}$$

They describe this as a *duality*, referring to $\{z_j\}$ as a particle system *dual to* $\{x_j\}$, and observe that the equations are symmetric under the exchange of particle positions and momenta if one also replaces the coupling constant γ by $-\gamma$.

Abanov et al specifically raise the question of how this duality might be related to bispectrality and to the action-angle duality discussed in Section 1.1. Following Wilson [**20**], we begin such an investigation by finding the particle systems governed by (1.1) as the eigenvalue dynamics associated to a flow on a manifold of matrix pairs satisfying a rank one condition. In fact, since the exchange of the coupling parameter γ with its additive inverse $-\gamma$ is apparent in the duality under investigation (see the last sentence of the previous paragraph), we will consider *two* different manifolds, one associated to γ and one to $-\gamma$, and algebraic maps between them. The main result below is Theorem 5.6 which identifies of one of these maps with the duality in (1.2) and (1.3). The paper then concludes with an outline of open questions and steps that ought to be taken to address them in future research.

1.3. Notation. Let $\omega, \gamma \in \mathbb{C}$ and $n \in \mathbb{N}$ be fixed non-zero parameters. Let $\mathbb{C}_{n\times n}$ denote the set of $n \times n$ matrices with complex valued entries. The symbol i will be used only to denote the imaginary number $\sqrt{-1}$ and $I = I_n$ represents the $n \times n$ identity matrix. The vector $e \in \mathbb{C}^n$ satisfies $e^\top = (1, \ldots, 1)$ so that $E = ee^\top \in \mathbb{C}_{n\times n}$ is the matrix with every entry equal to 1. The parameter g is to be considered a free variable taking non-zero values and having the interpretation of a coupling constant, so that whenever the particle system (1.1) is being considered, g will take either the value γ or $-\gamma$.

2. Matrix Pairs Satisfying a Rank One Condition

Let $hCM(n, g)$ denote the set of matrix pairs

$$hCM(n, g) = \{(X, L) \in \mathbb{C}_{n\times n} \times \mathbb{C}_{n\times n} \mid \operatorname{rank}(i[L, X] + gI_n) = 1\}.$$

For example, if $\{x_1, x_2, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n\}$ are $2n$ numbers with $x_j \neq x_k$ whenever $j \neq k$ then for the matrices X and L defined by

$$X_{jk} = x_j \delta_{jk} \qquad L_{jk} = \dot{x}_j \delta_{jk} + (1 - \delta_{jk}) \frac{ig}{x_j - x_k} \tag{2.1}$$

the matrix $i[L, X] + gI_n = gE$ is the square $n \times n$ matrix with every element equal to g. Hence, $(X, L) \in hCM(n, g)$.

We say $(X, L) \sim (\tilde{X}, \tilde{L})$ for two elements of $hCM(n, g)$ if there is a matrix $G \in GL(n)$ such that $\tilde{X} = GXG^{-1}$ and $\tilde{L} = GLG^{-1}$. Let $\overline{hCM}(n, g)$ be the quotient of $hCM(n, g)$ modulo this equivalence relation. Note, for example, that if $(X', L') \in hCM(n, g)$ and the eigenvalues of X' are distinct then there is a representative (X, L) of the equivalence class of (X', L') in $\overline{hCM}(n, g)$ in the canonical form (2.1).

3. Harmonic Dynamics Restricted to $hCM(n,g)$

For matrices $\mathcal{X}=(X,L)\in\mathbb{C}_{n\times n}\times\mathbb{C}_{n\times n}$ define $Q_{\mathcal{X}}=Q_{(X,L)}$ by the formula

$$Q_{\mathcal{X}}(t)=\cos(\omega t)X+\frac{1}{\omega}\sin(\omega t)L. \tag{3.1}$$

For given $n\times n$ matrices X and L it is easy to see that $Q(t)=Q_{\mathcal{X}}(t)$ is the unique solution to the initial value problem

$$\ddot{Q}(t)=-\omega^2 Q(t)\qquad Q(0)=X\qquad \dot{Q}(0)=L. \tag{3.2}$$

A central idea in integrable systems is that such simple motion in a high dimensional space (in this case, n^2-dimensional) can project to seemingly complicated dynamics when restricted to a lower dimensional manifold. For instance, one can find the dynamics of the particle system (1.1) by restricting this flow to $hCM(n,g)$, but first we must make sure such a restriction even makes sense.

Theorem 3.1. *Consider the vector field V on $\mathbb{C}_{n\times n}\times\mathbb{C}_{n\times n}$ defined by*

$$V(X,L)=(L,-\omega^2X)\qquad \mathcal{X}=(X,L)\in\mathbb{C}_{n\times n}\times\mathbb{C}_{n\times n}.$$

This vector field is tangent to $hCM(n,g)$ and induces the periodic flow $(Q_{\mathcal{X}}(t),\dot{Q}_{\mathcal{X}}(t))$ where $Q_{\mathcal{X}}$ is as defined in (3.1) *above.*

Proof. It is a direct consequence of (3.2) that $\frac{d}{dt}Q_{\mathcal{X}}\Big|_{t=0}=L$ and

$$\frac{d}{dt}\dot{Q}_{\mathcal{X}}\Big|_{t=0}=\ddot{Q}_{\mathcal{X}}(0)=-\omega^2Q_{\mathcal{X}}(0)=-\omega^2X.$$

Moreover, since

$$\begin{aligned}
[Q_{\mathcal{X}}(t),\dot{Q}_{\mathcal{X}}(t)] &= \left[\cos(\omega t)X+\frac{1}{\omega}\sin(\omega t)L,-\omega\sin(\omega t)X+\cos(\omega t)L\right]\\
&= [\cos(\omega t)X,\cos(\omega t)L]+\left[\omega^{-1}\sin(\omega t)L,-\omega\sin(\omega t)X\right]\\
&= \cos^2(\omega t)[X,L]-\sin^2(\omega t)[L,X]\\
&= (\cos^2(\omega t)+\sin^2(\omega t))[X,L]=[X,L]
\end{aligned}$$

it follows that if $\operatorname{rank}(i[L,X]+gI_n)=1$ then $(Q_{\mathcal{X}}(t),\dot{Q}_{\mathcal{X}}(t))$ is in $hCM(n,g)$ for all t. □

We are going to be interested in the dynamics of the eigenvalues induced by this flow and so further define

$$\mathcal{E}_{\mathcal{X}}=\{x_1(t),\dots,x_n(t)\}\qquad\text{where}\qquad \det(Q_{\mathcal{X}}(t)-\lambda I)=\prod_{j=1}^{n}(x_j(t)-\lambda). \tag{3.3}$$

It is important to note that $\mathcal{E}_{\mathcal{X}}=\mathcal{E}_{\mathcal{X}'}$ when $\mathcal{X}\sim\mathcal{X}'$ and so this map is well defined on $\overline{hCM}(n,g)$. (A less important observation is that, despite the indexing which is provided for notational convenience, the image of the map is defined only as an unordered set.)

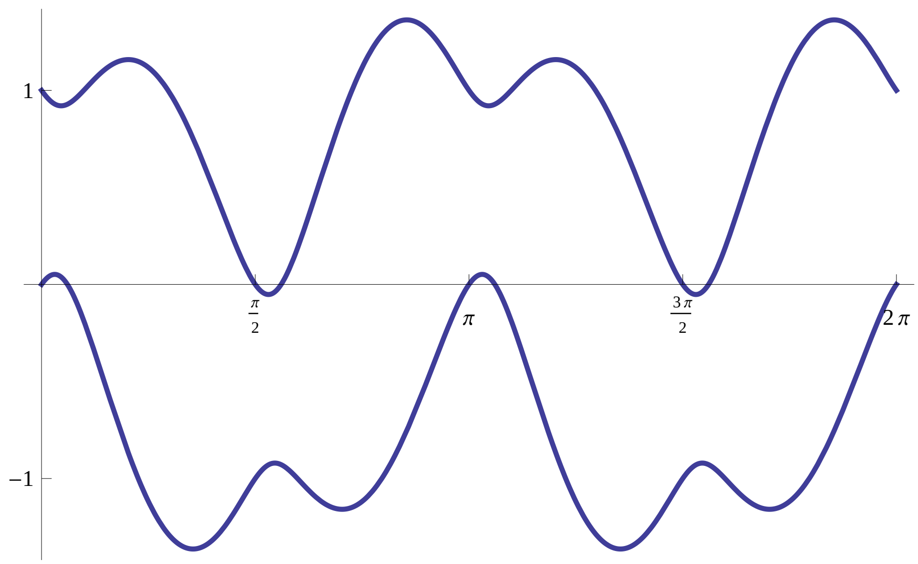

FIGURE 1. Dynamics in the real case beginning with an initial condition in canonical form (2.1). (See Example 3.3.)

PROPOSITION 3.2. *For* $\mathcal{X} = (X, L) \in hCM(n, \gamma)$ *the eigenvalues* $\mathcal{E}_{\mathcal{X}}$ *give the positions of harmonic Calogero particles with dynamics determined by* (1.1). *In particular, if* $\mathcal{E}_{\mathcal{X}} = \{x_1, \ldots, x_n\}$ *then*

$$\ddot{x}_j = -\omega^2 x_j - \sum_{k \neq j} \frac{2\gamma^2}{(x_k - x_j)^3} \qquad j = 1, \ldots, n \tag{3.4}$$

so long as $x_k \neq x_j$ *for* $k \neq j$*.*

This result is proved by Perelomov [**18**] (see the discussion of "systems of type V" spread throughout Chapter 3, and also references therein) in the real case beginning with (X, L) in the form (2.1). In that case, the particle positions remain distinct for all time. The same proof works in the complex case for the interval of time around $t = 0$ for which the x_k's are distinct. The general case then follows from the earlier observation that $\mathcal{E}_{\mathcal{X}}$ depends only on the class of $\mathcal{X}$ in $\overline{hCM}(n, g)$ and so as long as the eigenvalues of X are distinct one may begin with (generally complex valued) matrices in the canonical form (2.1).

EXAMPLE 3.3. A standard example in the case $n = 2$ would be to consider the canonical matrices X and L from (2.1) with $\gamma = \omega = 2$, $x_j = j-1$ and $\dot{x}_j = -(-1)^j$. Then the eigenvalues of the matrix

$$Q_{\mathcal{X}}(t) = \begin{pmatrix} \frac{1}{2}\sin(2t) & -i\sin(2t) \\ i\sin(2t) & \cos(2t) - \frac{1}{2}\sin(2t) \end{pmatrix}$$

are

$$x_j(t) = \frac{1}{16}\left(8\cos(2t) + (-1)^j\sqrt{64\cos^2(2t) - 32(2\sin(4t) + 5\cos(4t) - 5)}\right),\ j = 1, 2.$$

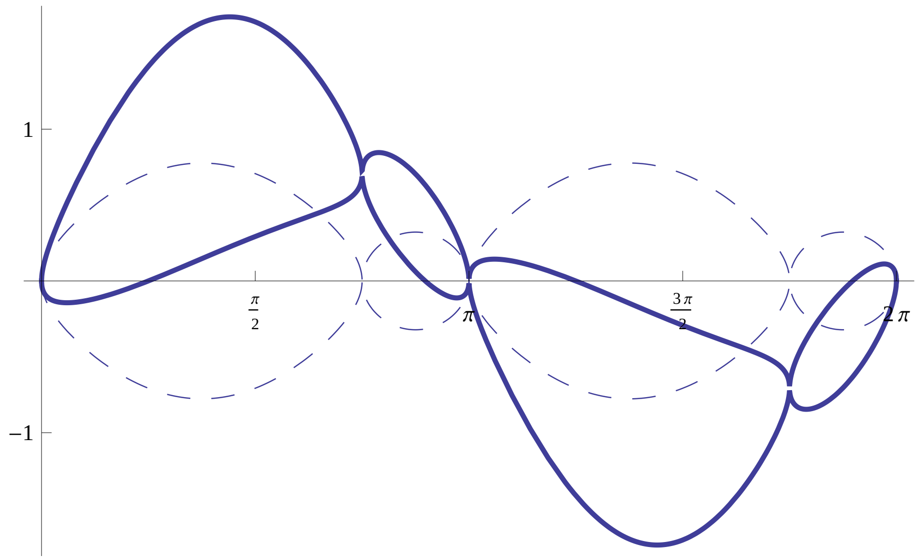

FIGURE 2. Dynamics exhibiting collisions from Example 3.4. The real parts of x_1 and x_2 are plotted with solid lines and the imaginary parts are plotted with dashed lines. Note that since $\mathcal{E}_{\mathcal{X}}$ is defined as an unordered set for each value of t, there is not a unique way to identify eigenvalues before and after a collision. In particular, one can either interpret the results of the collisions as a "bounce" in which the particles turn around after the interaction or as if the particles pass through each other.

We can check that they are governed by (1.1) by verifying that for $j \neq k$ one has $\ddot{x}_j + \frac{2\gamma^2}{(x_j - x_k)^3} + \omega^2 x_j = 0$. These are real valued functions exhibiting periodic dynamics and no collisions, as illustrated in Figure 1.

EXAMPLE 3.4. On the other hand, if

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad L = \begin{pmatrix} 1 & 1 \\ i\gamma & 1 \end{pmatrix}$$

then $\mathcal{X} = (X, L) \in hCM(n, \gamma)$ even though X is not diagonalizable. In the case $\gamma = \omega = 1$ the matrix $Q_{\mathcal{X}}(t)$ takes the form

$$Q_{\mathcal{X}}(t) = \begin{pmatrix} \sin(t) & \cos(t) + \sin(t) \\ i\sin(t) & \sin(t) \end{pmatrix}$$

with eigenvalues

$$x_j(t) = \sin(t) + (-1)^j \sqrt{i\sin^2(t) + i\sin(t)\cos(t)}.$$

Again, since $\ddot{x}_j + \frac{2\gamma^2}{(x_j - x_k)^3} + \omega^2 x_j = 0$ (when $x_1 \neq x_2$) the dynamics is determined by (1.1), but the functions now take complex values and collisions occur, as shown in Figure 2.

4. Some Algebraic Maps Taking $hCM(n,g)$ to $hCM(n,-g)$

Since the Hamiltonian (1.1) depends only on the square of the coupling constant γ, one should not be surprised to learn that solutions to the associated equations can be constructed from $hCM(n,-\gamma)$ as well as from $hCM(n,\gamma)$. This section clarifies the relationship by introducing some obvious algebraic maps between them and determining the dynamic significance.

THEOREM 4.1. *The function F_1 defined by the formula*

$$F_1(X,L) = (X,-L)$$

is a map from $hCM(n,g)$ to $hCM(n,-g)$. The eigenvalues $\mathcal{E}_{\mathcal{X}} = \{x_1,\ldots,x_n\}$ and $\mathcal{E}_{F_1(\mathcal{X})} = \{\tilde{x}_1,\ldots,\tilde{x}_n\}$ (suitably reordered) are related by the formula $x_j(t) = \tilde{x}_j(-t)$ for all values of t.

PROOF. For $n\times n$ matrices X and L, $i[L,X]+gI_n = -(i[-L,X]-gI_n)$ and so if $(X,L)\in hCM(n,g)$ the left-side is a matrix of rank equal to 1, which means that $i[-L,X]-gI_n$ is also a rank one matrix. Hence $(X,-L)\in hCM(n,-g)$. Moreover, it follows from the fact that cosine is an odd function and sine an even function that

$$\begin{aligned} Q_{\mathcal{X}}(t) &= \cos(\omega t)X + \frac{1}{\omega}\sin(\omega t)L \\ &= \cos(-\omega t)X - \frac{1}{\omega}\sin(-\omega t)L \\ &= \cos(\omega(-t))X + \frac{1}{\omega}\sin(\omega(-t))(-L) = Q_{F_1(\mathcal{X})}(-t) \end{aligned}$$

and so $\mathcal{E}_{\mathcal{X}}(t) = \mathcal{E}_{F_1(\mathcal{X})}(-t)$. □

THEOREM 4.2. *The function F_2 defined by the formula*

$$F_2(X,L) = (X^\top, L^\top)$$

is a map from $hCM(n,g)$ to $hCM(n,-g)$. The eigenvalues $\mathcal{E}_{\mathcal{X}} = \{x_1,\ldots,x_n\}$ and $\mathcal{E}_{F_2(\mathcal{X})} = \{\tilde{x}_1,\ldots,\tilde{x}_n\}$ (suitably reordered) are the same for all values of t.

PROOF. For $n\times n$ matrices X and L, $i[L,X]+gI_n = -(i[L^\top,X^\top]-gI_n)^\top$ and so if $(X,L)\in hCM(n,g)$ the left-side is a matrix of rank equal to 1, which means that $i[L^\top,X^\top]-gI_n$ is also a rank one matrix. Hence $(X^\top,L^\top)\in hCM(n,-g)$. Moreover, since $Q_{\mathcal{X}}(t) = Q^\top_{F_2(\mathcal{X})}(t)$ and the matrices related by transpose have the same eigenvalues, $\mathcal{E}_{\mathcal{X}} = \mathcal{E}_{F_2(\mathcal{X})}$. □

Combining these results with Proposition 3.2, we immediately have the corollary:

COROLLARY 4.3. *Let $\mathcal{X}\in hCM(n,\gamma)$ and let $\{x_1,\ldots,x_n\} = \mathcal{E}_{\mathcal{X}}$ be the corresponding particle system governed by* (1.1). *The matrix pair $F_2(\mathcal{X})\in hCM(n,-\gamma)$ corresponds to the very same particle state (reflecting the fact that the Hamiltonian is invariant under the change $\gamma\to-\gamma$), and the matrix pair $F_1(\mathcal{X})\in hCM(n,-\gamma)$ corresponds to a particle state with the same initial positions and orbits, but following the trajectories in the opposite direction (reflecting the invariance of the Hamiltonian under time-reversal).*

So $hCM(n,\gamma)$ and $hCM(n,-\gamma)$ produce the exact same set of solutions to the dynamical equations associated to (1.1). Nevertheless, rather than viewing this as an unnecessary redundancy, in order to reflect the change in sign of the coupling parameter associated to the duality, the next section will encode the duality in the form of another algebraic map from $hCM(n,\gamma)$ into $hCM(n,-\gamma)$.

5. Duality

Let $hCM^*(n,g)$ denote the open subset for which the matrix $\omega X + iL$ is invertible:

$$hCM^*(n,g) = \{(X,L) \in hCM(n,g) \mid \det(\omega X + iL) \neq 0\}.$$

For $\mathcal{X} \in hCM^*(n,g)$ we define the action of the maps $\mathfrak{d}_g^+$ and $\mathfrak{d}_g^-$ by the formula

$$\mathfrak{d}_g^+(X,L) = \big(X + g(\omega X + iL)^{-1}, L + i\omega g(\omega X + iL)^{-1}\big) \tag{5.1}$$

and

$$\mathfrak{d}_g^-(X,L) = F_2 \circ \mathfrak{d}_g^+. \tag{5.2}$$

The main result of this section will be to show that these definitions are closely related to the concept of duality in Abanov, Gromov and Kulkarni [**1**].

Lemma 5.1. *The domain and range of the maps $\mathfrak{d}_g^+$ and $\mathfrak{d}_g^-$ are given by*

$$\mathfrak{d}_g^\pm : hCM^*(n,g) \to hCM^*(n,\pm g).$$

Proof. Let $(X',L') = \mathfrak{d}_g^+(\mathcal{X})$ for $\mathcal{X} \in hCM(n,g)$. Then

$$\begin{aligned}
[X',L'] &= [X + g(\omega X + iL)^{-1}, L + i\omega g(\omega X + iL)^{-1}] \\
&= [X,L] + i\omega[g(\omega X + iL)^{-1}, g(\omega X + iL)^{-1}] \\
&\quad + [X, i\omega g(\omega X + iL)^{-1}] + [g(\omega X + iL)^{-1}, L] \\
&= [X,L] + 0 + [\omega X, ig(\omega X + iL)^{-1}] + [iL, ig(\omega X + iL)^{-1}] \\
&= [X,L] + 0 + ig[\omega X + iL, (\omega X + iL)^{-1}] = [X,L].
\end{aligned}$$

Hence, if $i[L,X] + gI$ is a rank one matrix, so is $i[L',X'] + gI$ which shows that $(X',L') \in hCM(n,g)$.

To see further that $(X',L') \in hCM^*(n,g)$ we note that due to convenient cancelation $\omega X' + iL' = \omega X + iL$ and the latter is invertible by assumption.

Finally, the claim in the case of the negative superscript follows from an application of Theorem 4.2. $\square$

In the following lemma, it is important to keep in mind that the action of $\mathfrak{d}_{-g}^+$ is given by formula (5.1) with both occurrences of g replaced by $-g$'s.

Lemma 5.2. *The maps $\mathfrak{d}_{-g}^-$ and $\mathfrak{d}_g^-$ are inverses: for any $\mathcal{X} = (X,L) \in hCM^*(n,g)$*

$$\mathfrak{d}_{-g}^- \circ \mathfrak{d}_g^-(\mathcal{X}) = \mathcal{X}.$$

Proof. For convenience, let us denote $\Theta = (\omega X + iL)^{-1}$ so that we can briefly write $\mathfrak{d}_g^-(\mathcal{X}) = (\hat{X}, \hat{L})$ as $\hat{X} = (X + g\Theta)^\top$ and $\hat{L} = (L + i\omega g\Theta)^\top$. Then

$$\begin{aligned}
\mathfrak{d}_{-g}^-(\mathfrak{d}_g^-(\mathcal{X})) &= \Big((\hat{X} - g(\omega\hat{X} + i\hat{L})^{-1})^\top, (\hat{L} - i\omega g(\omega\hat{X} + i\hat{L})^{-1})^\top\Big) \\
&= \big(X + g\Theta - g(\omega(X + g\Theta) + i(L + i\omega g\Theta))^{-1}, \\
&\quad L + i\omega g\Theta - i\omega g(\omega(X + g\Theta) + i(L + i\omega g\Theta))^{-1}\big) \\
&= \big(X + g\Theta - g(\omega X + \omega g\Theta + iL - \omega g\Theta^{-1}\ )^{-1}, \\
&\quad L + i\omega g\Theta - i\omega g(\omega X + \omega g\Theta + iL - \omega g\Theta)^{-1}\big) \\
&= \big(X + g\Theta - g(\omega X + iL)^{-1}, L + i\omega g\Theta - i\omega g(\omega X + iL)^{-1}\big) \\
&= (X + g\Theta - g\Theta, L + i\omega g\Theta - i\omega g\Theta) \\
&= (X, L)
\end{aligned}$$

□

Let us suppose we have fixed a choice of $\mathcal{X} = (X, L)$ in $hCM^*(n, g)$ written in the canonical form (2.1) and again denote $\mathfrak{d}_g^-(X, L) = (\hat{X}, \hat{L})$. The main goal of this section will be to relate the eigenvalues $\{z_j\}$ of $\hat{X}$ and the parameters x_j and $\dot{x}_j$ appearing in X and L. In order to do so, it is convenient to define the rational function

$$r(z) = \frac{d}{dz}\log\det(\hat{X} - zI)$$

which will appear only in the next two lemmas.

Lemma 5.3. *The function $r(z)$ can be written as $\displaystyle\sum_{j=1}^{n}\frac{1}{z - z_k}$ where z_k are the eigenvalues of $\hat{X}$.*

Proof. This is elementary, since any linear algebra student should know that $\det(\hat{X} - zI) = \prod_{k=1}^n (z_k - z)$ and any calculus student should know that

$$\frac{d}{dz}\log\prod_{k=1}^{n}(z_k - z) = \sum_{k=1}^{n}\frac{d}{dz}\log(z_k - z) = \sum_{k=1}^{n}\frac{-1}{z_k - z}.$$

□

Lemma 5.4. *For any eigenvalue x_j of X one can write $r(x_j)$ as:*

$$r(x_j) = \frac{\dot{x}_j}{ig} - \frac{\omega}{g}x_j + \sum_{k\neq j}\frac{1}{x_j - x_k}.$$

PROOF. First, we obtain a formula for $r(z)$ that does not involve determinants.

$$\begin{aligned}
r(z) &= \frac{d}{dz}\log\det(\hat{X}^\top - zI) \\
&\quad \text{(because the determinant is unaffected by transpose)} \\
&= \frac{d}{dz}\log\det(X + g(\omega X + iL)^{-1} - zI) \\
&= \frac{d}{dz}\log\det(\omega X + iL)\det(X + g(\omega X + iL)^{-1} - zI) \\
&\quad \text{(because the log derivative is unaffected by constant multiples)} \\
&= \frac{d}{dz}\log\det\big((\omega X + iL)(X + g(\omega X + iL)^{-1} - zI)\big) \\
&\quad \text{(because the determinant of a product is the product of the determinants)} \\
&= \frac{d}{dz}\log\det\left((\omega X + iL)X + gI - z(\omega X + iL)\right)) \\
&= \frac{d}{dz}\log\det\big(\omega X^2 + iLX + gI - z(\omega X + iL))\big) \\
&= \frac{d}{dz}\log\det\big(\omega X^2 + (gE + iXL - gI) + gI - z(\omega X + iL))\big) \\
&\quad \text{(using the rank one relationship } i[L,X] + gI = gE\text{)} \\
&= \frac{d}{dz}\log\det\left((X - zI)(\omega X + iL) + gE\right)) \\
&= \frac{d}{dz}\log\big(1 + ge^\top(\omega X + iL)^{-1}(X - zI)^{-1}e\big)\det(X - zI)\det(\omega X + iL) \\
&\quad \text{(using the identity that } \det(A + uv^\top) = \det(A)(1 + v^\top A^{-1}u)\text{)} \\
&= \frac{d}{dz}\log\big(1 + ge^\top(\omega X + iL)^{-1}(X - zI)^{-1}e\big)\prod_{k=1}^{n}(x_k - z) \\
&\quad \text{(since } \det(X - zI) = \textstyle\prod(x_i - z) \text{ and the other determinant is constant)}
\end{aligned}$$

Now, define $f(z) = \big(1 + ge^\top(\omega X + iL)^{-1}(X - zI)^{-1}e\big)\prod_{k=1}^{n}(x_k - z)$ to be the function whose logarithmic derivative is $r(z)$. Clearly, $r(x_j) = f'(x_j)/f(x_j)$ and so we only need to determine the numerator and denominator in this expression. We will write each of them in terms of the elements of the vector

$$\eta^\top = (\eta_1\ \eta_2\ \cdots\ \eta_n) = ge^\top(\omega X + iL)^{-1}.$$

Matrix multiplication yields that

$$f(z) = \prod_{k=1}^{n}(x_k - z) + \sum_{k=1}^{n}\eta_k\prod_{\alpha\neq k}(x_\alpha - z)$$

and

$$f'(z) = -\sum_{k=1}^{n}\prod_{\alpha\neq k}(x_\alpha - z) - \sum_{k=1}^{n}\eta_k\sum_{\alpha\neq k}\prod_{\beta\neq\alpha,k}(x_\beta - z).$$

Upon evaluation at $z = x_k$ we obtain

$$f(x_k) = \eta_j\prod_{\alpha\neq j}(x_\alpha - x_j)$$

and

$$f'(x_j) = -\prod_{\alpha\neq j}(x_\alpha - x_j) - \sum_{\alpha\neq j}(\eta_\alpha + \eta_j)\prod_{\beta\neq\alpha,j}(x_\beta - x_j).$$

Consequently,

$$r(x_j) = \frac{f'(x_j)}{f(x_j)} = -\frac{1}{\eta_j}\left(1 + \sum_{\alpha\neq j}\frac{\eta_\alpha}{x_\alpha - x_j}\right) + \sum_{\alpha\neq j}\frac{1}{x_j - x_\alpha}.$$

By definition, we have that $\eta^\top(\omega X + iL) = ge$. Considering only the j^{th} column give us that

$$\eta_j(\omega x_j + i\dot{x}_j) - \sum_{\alpha\neq j}\frac{g\eta_\alpha}{x_\alpha - x_j} = g$$

or equivalently that

$$\sum_{\alpha\neq j}\frac{g\eta_\alpha}{x_\alpha - x_j} = \frac{\eta_j}{g}(\omega x_j + i\dot{x}_j) - 1.$$

Substituting this into our last expression for $r(x_j)$ we obtain the desired formula. $\square$

Equating the expression for $r(x_j)$ provided by the preceding two lemmas, we determine the following identity relating the eigenvalues of X and $\hat{X}$:

Corollary 5.5. *Let $\mathcal{X} = (X, L) \in hCM(n, g)$ be written in the canonical form* (2.1) *and let $(\hat{X}, \hat{Z}) = \mathfrak{d}_g^-(\mathcal{X}) \in hCM(n, -g)$. The eigenvalues $\{z_k\}$ of $\hat{X}$ satisfy*

$$\sum_{k=1}^{n}\frac{1}{x_j - z_k} = \frac{\dot{x}_j}{ig} - \frac{\omega}{g}x_j + \sum_{k\neq j}\frac{1}{x_j - x_k}$$

for each $j = 1, \ldots, n$.

We now have all of the ingredients necessary to prove the main result:

Theorem 5.6. *Let $\mathcal{X} = (X, L) \in hCM^*(n, \gamma)$ such that X has distinct eigenvalues and*

$$\hat{\mathcal{X}} = (\hat{X}, \hat{L}) = \mathfrak{d}_\gamma^-(\mathcal{X}) \in hCM^*(n, -\gamma).$$

Then the particle systems $\mathcal{E}_{\mathcal{X}} = \{x_1, \ldots, x_n\}$ and $\mathcal{E}_{\hat{\mathcal{X}}} = \{z_1, \ldots, z_n\}$ associated to each are dual systems satisfying the equations (1.2) *and* (1.3)*.*

Proof. Without loss of generality, we may assume that (X, L) are written in canonical form (2.1) since the statement depends only on the class of $\mathcal{X}$ in $\overline{hCM}(n, \gamma)$. Then Corollary 5.5 immediately implies (1.2) upon the substitution $g = \gamma$.

To obtain (1.3) (with $m = n$), we apply Corollary 5.5 to $\hat{X}$, which requires using $-\gamma$ for g, writing z_k in place of x_k (since these are the eigenvalues of $\hat{X}$) and x_k in place of z_k (since by Lemma 5.2, $\mathcal{X}$ is the image of $\hat{\mathcal{X}}$ under the map $\mathfrak{d}_{-\gamma}^-$). $\square$

Example 5.7. Let $n = \omega = \gamma = 2$ and consider the point $\mathcal{X} = (X, L) \in hCM(2, 2)$ in canonical form given by

$$X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad L = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}.$$

Then $\mathcal{E}_{\mathcal{X}} = \{x_1(t), x_2(t)\}$ where

$$x_j(t) = \frac{1}{4}\left(2\cos(2t) + (-1)^j\sqrt{10-6\cos(4t)}\right)$$

and so

$$x_1(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0 \text{ and } x_2(0) = 1. \tag{5.3}$$

Computing $\hat{\mathcal{X}} = \mathfrak{d}^-_\gamma(\mathcal{X}) = (\hat{X}, \hat{L})$ we find

$$\hat{X} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \text{and} \qquad \hat{L} = \begin{pmatrix} 2i & 4i \\ -4i & 0 \end{pmatrix}$$

which are not in canonical form but satisfy $i[\hat{L}, \hat{X}] - 2I = 2E$ and hence $\hat{\mathcal{X}}$ is in $hCM(2, -2)$. Then $\mathcal{E}_{\hat{\mathcal{X}}} = \{z_1(t), z_2(t)\}$ where

$$z_j(t) = \frac{1}{4}\left(2i\sin(2t) + 4\cos(2t) + (-1)^j\sqrt{-32i\sin(4t) - 38\cos(4t) + 22}\right)$$

and so

$$z_1(0) = 1 - i \qquad z_2(0) = 1 + i \qquad \dot{z}_1(0) = 4 + i \qquad \text{and} \qquad \dot{z}_2(0) = -4 + i. \tag{5.4}$$

In agreement with the claim of Theorem 5.6, the values given in (5.3) and (5.4) satisfy both duality equations (1.2) and (1.3).

6. Closing Remarks

Abanov et all [**1**] asked whether this other sort of duality of the Calogero system in a harmonic potential might have an interpretation in terms of bispectrality as does the action-angle duality of particle systems in all known instances. This paper has not answered this question, but rather has taken some steps towards that goal by recasting the problem in the language of manifolds of matrices satisfying rank one conditions that has proved to be useful in the other situation. The main result has been to present a closed algebraic formula associating a dual state to a large open subset of initial states of the system. (The paper [**1**] does not give such a formula, but it does give a formula for matrices in canonical form corresponding to the dual depending on *implicitly* determined parameters z_k and $\dot{z}_k$. The matrices they give are in $hCM(n, \gamma)$, whereas the ones given here are not in canonical form and are found in $hCM(n, -\gamma)$. Of course, their conjugacy classes are related by the involution F_2.)

These results are encouraging and suggest that further investigation in this regard should be undertaken. Following are some specific topics of interest which were not considered here and speculation about how they might be addressed in the future.

- The construction presented above always produces a dual system with the same number of particles as the original system. In other words, this paper considers only the case $m = n$. One of the most interesting things in [**1**] is that they allow different numbers of particles in the original system and its dual.
- Another interesting aspect of [**1**] which was not considered here is the connection to soliton equations. Writing differential Lax operators and wave functions for a soliton equation in terms of the elements of $hCM(n, \gamma)$ is clearly a step that should be taken in order to address the question raised

by Abanov, Gromov and Kulkarni regarding a possible connection to bispectrality. I would like to suggest, without any evidence at this point, that the correct approach may be to construct $2n \times 2n$ matrices which satisfy rank *two* conditions and have the matrices from this paper as blocks. Associated to these larger matrices might be solutions to the two-component KP hierarchy in which the duality map has a natural interpretation, as it did in the case of action-angle duality. Moreover, considering the blocks as being rectangular rather than square may provide a means to address the duality between different numbers of particles.

- The construction of the dual map presented above is quite general, but not universal. In particular, it assumes the invertibility of $\omega X + iL$. It would be interesting to know whether this restriction is actually inherent in the duality defined by Abanov et al, and if not how this algebraic formula could be extended to the general case.
- The operators $\omega X \pm iL$ are important in the general theory of the harmonic Calogero system [**1, 18**]. As can be seen in the proofs of Theorem 4.2 and Lemma 5.1, it turns out to be convenient that $\omega X + iL = \omega X' + iL'$ when $(X', L') = \mathfrak{d}_g^+(X, L)$ (a). There may be greater significance to the fact that this matrix is an invariant of the dual map.
- Finally, it may also prove interesting to investigate the standard action-angle duality and the more familiar involution $(X, L) \mapsto (L^\top, X^\top)$ on $hCM(n, g)$ to determine whether they lead to any new bispectral operators.

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