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INTEGRABLE SYSTEMS AND RANK-ONE CONDITIONS FOR RECTANGULAR MATRICES

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We give a determinantal formula for tau functions of the KP hierarchy in terms of rectangular *constant matrices A, B, and C satisfying a rank-one condition. This result is shown to generalize and unify many previous results of different authors on constructions of tau functions for differential and difference integrable systems from* square *matrices satisfying rank-one conditions. In particular, its explicit special cases include Wilson's formula for tau functions of the rational KP solutions in terms of Calogero–Moser Lax matrices and our previous formula for the KP tau functions in terms of almost-intertwining matrices.*

Keywords: KP hierarchies, solitions, Calogero–Moser matrices, rank-one conditions

1. Introduction

In many recent papers by different authors, determinantal formulas have been used to transform constant square matrices satisfying a rank-one condition into tau functions for integrable systems. In particular, we recall Wilson's result [1] that $n\times n$ matrices X and Z satisfying the "almost-canonically conjugate" condition

$$\operatorname{rank}([X,Z]+I)=1 \tag{1}$$

produce tau functions for rational solutions to the KP hierarchy by the formula

$$\tau(\vec{t}\,)=\det\biggl(X+\sum_{i=1}^{\infty} it_iZ^{i-1}\biggr) \tag{2}$$

(also see [2]–[4]). This result can be interpreted as a relation between the KP hierarchy and the Calogero–Moser particle system and is therefore similar to the relation between KdV solitons and the Ruijsenaars–Schneider particle system [5], [6]. In that case, the Lax matrices X and Z for this particle system satisfy the rank-one condition $\operatorname{rank}(XZ+ZX)=1$, and the formula

$$\tau(\vec{t}\,)=\det\biggl(\exp\biggl(\sum_{i=0}^{\infty}t_{2i+1}Z^{2i+1}\biggr)X\exp\biggl(\sum_{i=0}^{\infty}t_{2i+1}Z^{2i+1}\biggr)+I\biggr) \tag{3}$$

gives the tau function for a multisoliton solution of the KdV hierarchy.

Both soliton and rational solutions were produced by a formula in our previous paper [7], where we showed that square matrices X, Y, and Z satisfying the "almost-intertwining" condition

$$\operatorname{rank}(XZ-YX)=1 \tag{4}$$

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produce KP tau functions via the formula

$$\tau(\vec{t}\,) = \det\left(X \exp\left(\sum_{i=1}^{\infty} t_i Z^i\right) + \exp\left(\sum_{i=1}^{\infty} t_i Y^i\right)\right). \tag{5}$$

Previous results have also demonstrated the usefulness of matrices satisfying these rank-one equations in constructing solutions of difference equations. In particular, it was shown in [8] that for matrices X and Z satisfying (1), the eigenvalues x_i^m of the matrix

$$\mathbf{X}(m) = -\eta X(\lambda_1 - Z) - m\eta(\lambda_2 - Z)^{-1}(\lambda_1 - Z) \tag{6}$$

(where η and λ_i are arbitrarily selected constants) satisfy the rational nested Bethe ansatz equations

$$\prod_{k=1}^{n} \frac{(x_j^m - x_k^{m-1})(x_j^m - x_k^m + \eta)(x_j^m - x_k^{m+1} - \eta)}{(x_j^m - x_k^{m-1} + \eta)(x_j^m - x_k^m - \eta)(x_j^m - x_k^{m+1})} = -1 \quad \forall 1 \le j \le n. \tag{7}$$

Moreover, we announced at NEEDS 2001 (as an immediate consequence of [7]) that the formula

$$\tau_l^{m,n} = \det\bigl[X(c_1 - Z)^l (c_2 - Z)^m (c_3 - Z)^n + (c_1 - Y)^l (c_2 - Y)^m (c_3 - Y)^n W\bigr] \tag{8}$$

gives a solution to the Hirota bilinear difference equation

$$(c_2 - c_3)\tau_{l+1}^{m,n}\tau_l^{m+1,n+1} - (c_1 - c_3)\tau_l^{m+1,n}\tau_{l+1}^{m,n+1} + (c_1 - c_2)\tau_l^{m,n+1}\tau_{l+1}^{m+1,n} = 0, \tag{9}$$

when X, Y, and Z satisfy (4) (for an arbitrary $n \times n$ matrix W).

In addition, two formulas have recently been published for Baker–Akhiezer functions of the q-KP hierarchy in terms of matrices satisfying q-variants of these "almost" operator identities. In particular, the formula

$$\psi(x, z) = \frac{\det(xzI + xqZ - zX - XZ - I)}{\det(xI - X)\det(zI + qZ)} e^{xz} \tag{10}$$

in the case where $\operatorname{rank}(XZ - qZX + I) = 1$ is found in [9]. Similarly, in [10], there is the formula

$$\psi(x, z) = \frac{\det(qZX - xZ - zX + zxI)}{\det(ZX - xZ - zX + zxI)} e_q^{xz} \tag{11}$$

in the case where $\operatorname{rank}(XZ - qZX) = 1$. In both cases, the interest in these particular Baker–Akhiezer functions lies in their bispectrality (cf. [11]).

In addition to the obvious similarities, some explicit connections have been drawn between these results. For example, formula (6) is easily derived from (2) (although this is not how it was derived in [8]), formula (5) allows constructing rational solutions like (2) and soliton solutions, and Eq. (3) can be viewed as a special case of (5). The formulas pertaining to the q-KP hierarchy must be related to the others by the correspondence in [9] (cf. [12]) between tau functions and wave functions of the q-KP and KP hierarchies. But clearly lacking is a general framework in which all of these different formulas appear as special cases. Our goal in this paper is to provide such a framework by generalizing the formulas to the case of *rectangular* matrices.

2. Main Results

Let A and C be full-rank $n\times N$ matrices and let B be an $N\times N$ matrix for $N > n$. For convenience, we assume that the nondegeneracy condition $\det(AC^{\mathrm{T}}) \neq 0$ is satisfied.

We let L be the subspace in $\mathbb{C}^N$ spanned by the rows of A and $L^\perp$ be its orthogonal complement with respect to the standard bilinear form $\langle x, y\rangle = \sum_{k=1}^N x_i y_i$. We fix a basis $u_1, \dots, u_{N-n}$ in $L^\perp$ and define U to be the $(N-n)\times N$ matrix with the rows $u_1, \dots, u_{N-n}$. As in [7], we let $g(x) = \sum_{i=1}^\infty t_i x^i$ be a power series in x with coefficients that depend on the time variables $\vec{t} = (t_1, t_2, \dots)$ of the KP hierarchy.

Theorem 1. *If*

$$\mathrm{rank}(ABU^{\mathrm{T}}) \leq 1, \tag{12}$$

then

$$\tau_l^{m,n} = \det\bigl[A(c_1 I - B)^l (c_2 I - B)^m (c_3 I - B)^n C^{\mathrm{T}}\bigr] \tag{13}$$

is a solution to Hirota bilinear difference equation (9) *and*

$$\tau(\vec{t}\,) = \det[Ae^{g(B)}C^{\mathrm{T}}] \tag{14}$$

is a tau function of the KP hierarchy.

Proof. It is known (see, e.g., [13]) that to prove that $\tau(\vec{t}\,)$ is a KP tau function it suffices to prove that $\tau_l^{m,n} := \tau(\vec{t} - l[c_1^{-1}] - m[c_2^{-1}] - n[c_3^{-1}])$ solves the Hirota bilinear difference equation for all values of the parameters.[1] Hence, our method is to prove the second claim above by proving the first. We note that it suffices to prove that the Hirota bilinear difference equation is satisfied when all the discrete "times" $l, m, n = 0$. This is because $\tau_l^{m,n} = \hat{\tau}_0^{0,0}$ if $\hat{\tau}$ is the tau function corresponding to the same choice of A and B but with a different C (multiplied by the transpose of $(c_1 I - B)^l \cdots (c_3 - B)^n$ from the right). Because the condition in the claim depends only on a property of A and B, it therefore suffices to consider the restricted version of the equation

$$\begin{aligned} &(c_2 - c_3)\tau\bigl(\vec{t} - [c_1^{-1}]\bigr)\tau\bigl(\vec{t} - [c_2]^{-1} - [c_3^{-1}]\bigr) - \\ &\quad - (c_1 - c_3)\tau\bigl(\vec{t} - [c_2^{-1}]\bigr)\tau\bigl(\vec{t} - [c_1^{-1}] - [c_3^{-1}]\bigr) + \\ &\qquad + (c_1 - c_2)\tau\bigl(\vec{t} - [c_3^{-1}]\bigr)\tau\bigl(\vec{t} - [c_1^{-1}] - [c_2^{-1}]\bigr) = 0. \end{aligned}$$

We reduce this equation to the identity proved in [7],

$$h_1(c_1)h_2(c_2, c_3) - h_1(c_2)h_2(c_1, c_3) + h_1(c_3)h_2(c_1, c_2) \equiv 0, \tag{15}$$

where $h_1(c_1) = \det[c_1 - P]$ and $h_2(c_1, c_2) = \det\bigl[(c_1 - P)(c_2 - P) + Q\bigr]$ with $n\times n$ matrices P and Q and $\mathrm{rank}\, Q \leq 1$.

First, we let V^{T} be any right inverse of A and define $G = [V^{\mathrm{T}}U^{\mathrm{T}}]$, $\widehat{B} = G^{-1}BG$, $\widehat{C}^{\mathrm{T}} = G^{-1}C^{\mathrm{T}}$, and $M = M(\vec{t}\,) = [I_n 0]e^{g(\widehat{B})}\widehat{C}^{\mathrm{T}}$. Then $AG = [I_n 0]$ and $\tau(\vec{t}\,) = \det\bigl[M(\vec{t}\,)\bigr]$. We also note that $G^{-1} = \begin{bmatrix} A \\ * \end{bmatrix}$.

[1] As usual, the Miwa shift $\vec{t} + c[z]$ is defined as

$$\vec{t} + c[z] = \biggl(t_1 + cz, t_2 + \frac{cz^2}{2}, t_3 + \frac{cz^3}{3}, \dots\biggr).$$

The Miwa shift $\tau\big(\vec{t} - [c^{-1}]\big)$ can be computed as

$$\tau\big(\vec{t} - [c^{-1}]\big) = \det\big[[I_n 0] e^{g(\widehat{B})} e^{\ln(I_N - c^{-1}\widehat{B})} \widehat{C}^{\,\mathrm{T}}\big] = $$
$$= c^{-n} \tau(\vec{t}\,) \det\big[c - [I_n 0]\widehat{B} e^{g(\widehat{B})} \widehat{C}^{\,\mathrm{T}} M(\vec{t}\,)^{-1}\big]$$

and, similarly,

$$\tau\big(\vec{t} - [c_1^{-1}] - [c_2^{-1}]\big) = c_1^{-n} c_2^{-n} \tau(\vec{t}\,) \det\big[[I_n 0](c_1 - \widehat{B})(c_2 - \widehat{B}) e^{g(\widehat{B})} \widehat{C}^{\,\mathrm{T}} M(\vec{t}\,)^{-1}\big].$$

It is then easy to verify that the left-hand side of the Hirota bilinear difference equation is proportional to the left-hand side of (15) if we define $P = [I_n 0]\widehat{B} e^{g(\widehat{B})} \widehat{C}^{\,\mathrm{T}} M^{-1}$ and $Q = [I_n 0](\widehat{B})^2 e^{g(\widehat{B})} \widehat{C}^{\,\mathrm{T}} M^{-1} - P^2$. It therefore suffices to show that $\operatorname{rank} Q \le 1$. We rewrite Q as

$$Q = [I_n 0]\widehat{B}\big(I - e^{g(\widehat{B})} \widehat{C}^{\,\mathrm{T}} M^{-1} [I_n 0]\big)\widehat{B} e^{g(\widehat{B})} \widehat{C}^{\,\mathrm{T}} M^{-1}$$

and note that $e^{g(\widehat{B})} \widehat{C}^{\,\mathrm{T}} = \begin{bmatrix} M \\ * \end{bmatrix}$. Therefore,

$$I - e^{g(\widehat{B})} \widehat{C}^{\,\mathrm{T}} M^{-1} [I_n 0] = \begin{bmatrix} 0 & 0 \\ * & I_{N-n} \end{bmatrix},$$

which implies that

$$[I_n 0]\widehat{B}\big(I - e^{g(\widehat{B})} \widehat{C}^{\,\mathrm{T}} M^{-1} [I_n 0]\big) = \left([I_n 0]\widehat{B} \begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right) [* I_{N-n}].$$

But the factor in parentheses is equal to ABU^{T} and, by our assumption, has a rank less than or equal to one. Therefore, the same is true for Q, which finishes the proof.

Remark. We emphasize that rank-one condition (12) is independent of the choice of bases in the subspaces L and $L^{\perp}$ that correspond to the matrix A.

Because we have constructed a solution to the KP hierarchy, it is interesting to know where the corresponding solutions lie in the Sato–Segal–Wilson[2] Grassmannian [14], [15]. This can be resolved by studying the associated Baker–Akhiezer function $\psi(x, z)$. In particular, it is known that the point $W \in \mathrm{Gr}$ is the subspace spanned by ψ and its x-derivatives evaluated at $x = 0$,

$$W = \big\langle \psi(0, z), \psi_x(0, z), \psi_{xx}(0, z), \dots \big\rangle.$$

Below, we show that there exist polynomials $p(z)$ and $q(z)$ such that $p(z)H_+ \subset W \subset q^{-1}(z)H_+$, which shows by definition that W is in the sub-Grassmannian $\mathrm{Gr}^{\mathrm{rat}}$ [16], [17].

[2] We change the form of the notation in [15] and therefore $H = H_+ \oplus H_-$ is the Hilbert space of Laurent series in z expressed as a direct sum such that H_+ is generated by nonnegative powers of z and H_- by negative powers. But we do not apply an analytic restriction to H. Then, as is well-known in theory, there exists a bijection of Grassmannian points of the subspaces W "corresponding" to H_+ and the tau functions given by the determinant of the map of the projection from $e^{-g(z)}W$ to H_+.

Theorem 2. *The stationary Baker–Akhiezer function corresponding to the tau function given in* (14) *is*

$$\psi(x,z)=\frac{\det\bigl(Ae^{xB}(zI-B)C\bigr)}{z^N\det(Ae^{xB}C)}e^{xz}. \tag{16}$$

As a consequence, we can determine that this solution corresponds to a point in the sub-Grassmannian $\mathrm{Gr}^{\mathrm{rat}}$ *of rank-one KP solutions with rational spectral curves.*

Proof. The formula for $\psi(\vec{t},z)$ follows from the well-known formula for the time-dependent wave function [15],

$$\psi(\vec{t},z)=\frac{\tau\bigl(\vec{t}-[z^{-1}]\bigr)}{\tau(\vec{t}\,)}e^{g(z)}.$$

Evaluated at $\vec{t}=(x,0,0,0,\dots)$, this simplifies to formula (16) because the coefficients of $-[x]$ are precisely the coefficients in the power series for $\log(1-x)$.

Another way to identify the subspace W is by its duality with the dual Baker–Akhiezer function ψ^*, which, by arguments similar to those above, can be shown to have the property that $p(z)\psi^*$ is nonsingular in z for $p(z)=\det(zI-B)$. Hence, the inner product of any polynomial in $p(z)H_+$ with ψ^* (computed as the path integral of the product around S^1) is zero. Consequently, $p(z)H_+\subset W$, and we see that W is in $\mathrm{Gr}^{\mathrm{rat}}$.

In the case of solutions corresponding to rational spectral curves [15]–[18], it has frequently been found useful to identify a solution instead by the finite-dimensional space of finitely supported distributions in z that annihilate the Baker–Akhiezer function. A consequence of the role of the characteristic polynomial $p(z)$ in the above proof is that the finitely supported distributions in z that annihilate $z^N\psi(x,z)$ are supported at the eigenvalues of B with the highest derivative taken bounded by the algebraic multiplicity of the eigenvalue.

3. Special Cases

As a corollary to the main theorem above, we give the following generalization of our theorem in [7].

Theorem 3. *Let* X *be an* $n\times(N-n)$ *matrix,* Y *be an* $n\times n$ *matrix, and* Z *be an* $(N-n)\times(N-n)$ *matrix such that*

$$\operatorname{rank}(XZ-YX)\le 1. \tag{17}$$

Then (14) *is a tau function of the KP hierarchy, where* $A=[XI_n]$, B *is the block diagonal matrix* $B=\operatorname{diag}[Z,Y]$*, and* C *is an arbitrary full-rank* $n\times(N-n)$ *matrix.*

To see that this is so, we note that U in (12) can be chosen as $[-I_nX^{\mathrm{T}}]$. We then find that condition (12) is given by (17). This generalizes (4) because it was assumed there that $N=2n$ and X is a square matrix. In the latter case, we can define C as $C=[I_nI_n]$, which transforms (14) into a tau function $\tau(\vec{t}\,)=\det[Xe^{g(Z)}+e^{g(Y)}]$ that coincides with (5).

In fact, combining the discussion in the previous paragraph with the observation that the distributions annihilating the Baker–Akhiezer function are supported at the eigenvalues of B with the order bounded by the algebraic multiplicity confirms the conjecture in [7] that the same is true for the eigenvalues of Y and Z.

Moreover, we can also rederive Wilson's formula (2) and the "almost-canonically conjugate" rank-one condition (1) as a special case of our main result. We consider the case where $N=2n$, A, C, and U

are defined as above, but B is chosen as $B = \begin{bmatrix} Z & 0 \\ I_n & Z \end{bmatrix}$. Then $ABU^{\mathrm{T}} = -\bigl([X, Z] + I_n\bigr)$, and (12) therefore coincides with (1). Moreover, in this case,

$$e^{g(B)} = \begin{bmatrix} e^{g(Z)} & 0 \\ g'(Z)e^{g(Z)} & e^{g(Z)} \end{bmatrix},$$

and tau function (14) becomes $\tau(\vec{t}\,) = \det\bigl[e^{g(Z)}\bigr]\det\bigl[X + g'(Z)\bigr]$, which is gauge equivalent to the one in (2).

In conclusion, although many of the details are yet to be fully explored, the formula that we proved above allows considering many different results relating rank-one conditions and tau functions in a unified context for the first time. In a future paper, we plan to address questions of the relation between the geometry of the space of matrices that we utilize and the geometry of the Grassmannian, reductions to finite-dimensional Hamiltonian (particle) systems, and the case where A, B, and C are taken as infinite-dimensional operators.

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