Test Results / Odds and Ends

- The test results could have been better, but they were good enough to encourage me that things will continue to improve throughout the semester. Let’s keep improving our ability to work with abstractions through proofs, true/false questions, etc.
- Just to reiterate the definition of “limit” and its meaning, let’s look at the first question and what it means when we get $N > -1 + 1/\epsilon$.
- As I keep telling you, intuition can fail us. I saw a good non-mathematical example of that in a magazine the other day. You know those “scared straight” programs which show juvenile delinquents what could face them in prison? They are supposed to convince them to go straight, but studies have now shown that the opposite is actually true! Kids who get that glimpse of jail life are more likely to commit crimes later. It is not clear why (do they idolize the guys in jail? does it lower their self-esteem to go through the program?) but when evidence contradicts our intuition, I think we’ve got to go with evidence.
- Here is a mathematical example for you. We’ve seen functions that were continuous only at $x = 0$, everywhere except at $x = 0$, and one that was continuous nowhere. However, I bet you would not believe that there is a function which is continuous only where this function is!

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = p/q \text{ is a rational number written in simplest form} \end{cases}$$

This function is continuous at every irrational number and discontinuous at every rational number!

Uniform Continuity

- Suppose the function $f : D \rightarrow \mathbb{R}$ is continuous. Then if $\{u_n\}$ and $\{v_n\}$ are sequences in $D$ which converge to the same point in the domain of $f$ whose entries converge to the same point in the domain of $D$, then the sequence $\{f(u_n) - f(v_n)\}$ will converge to zero. However, if we weaken the requirement from saying that the two sequences converge to some point in $D$ to merely saying that they “come together”, then this need not be the case, as the next example illustrates:

Let $u_n = n$ and $v_n = n + 1/n$. Note that $u_n - v_n = 1/n$, so these do get closer and closer. However, if we put them in the function $f(x) = x^2$ and subtract we get $f(u_n) - f(v_n) = 2 + 1/n^2$. Even though the sequences seem to be converging into each other, the values of the function at those points never gets closer than 2 units apart!

- We will view this as a property of the function $f$. The nicest functions, the ones that are not just continuous but somehow even more than just continuous are ones for which this can never happen. Here’s the official definition:
**Definition:** A function $f : D \to \mathbb{R}$ is said to be **uniformly continuous** provided that whenever $\{u_n\}$ and $\{v_n\}$ are sequences in $D$ such that

$$\lim_{n \to \infty} u_n - v_n = 0$$

then

$$\lim_{n \to \infty} f(u_n) - f(v_n) = 0$$

also.

- There are three ways a function can fail to be uniformly continuous.
  - It could not be continuous on $D$. (If a function is not continuous, it cannot be uniformly continuous. To see that this is true, let $\{a_n\}$ be a sequence in $D$ converging to $a \in D$ such that $\{f(a_n)\}$ does not converge to $f(a)$, which must exist because we said the function is not uniformly continuous. Then $u_n = a_n$ and $v_n = a$ show that the conditions of the definition fail.)
  - It could be continuous but become very “steep” so that there are points on the graph which are far apart vertically even though their $x$-coordinates are close, as does the function $f(x) = x^2$ in the first example above. (We have not yet discussed derivatives, but we will see later that a connection exists between $f'(D)$ being bounded and uniform continuity.)
  - Non-uniform continuity can also follow if a function has a hole in its domain. For example, the function

  $$f(x) = \begin{cases} 
  0 & x < 1 \\
  1 & x > 1 
  \end{cases}$$

  is continuous on the domain $[0, 1) \cup (1, 2]$. The sequences $u_n = 1 - \frac{1}{n}$ and $v_n = 1 + \frac{1}{n}$ though show that it is not uniformly continuous since $u_n - v_n = -2/n$ (which does go to zero) while $f(u_n) - f(v_n) = 1$ (which does not).

- Notice that to show that a function is not uniformly continuous requires finding only one counter-example. For instance, our examples above each show that a function is not uniformly continuous by giving one example. On the other hand, showing that something is uniformly continuous requires showing that the second equation above is true for any sequences that satisfy the first equation! The following example illustrates how you can show that a function is uniformly continuous by considering all possible sequences $\{u_n\}$ and $\{v_n\}$:

  - The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = mx + b$ for some fixed numbers $m$ and $b$ is uniformly continuous. To see that this is true, let $\{u_n\}$ and $\{v_n\}$ be any sequences such that $\lim u_n - v_n = 0$. Then,

    $$\lim_{n \to \infty} f(u_n) - f(v_n) = \lim_{n \to \infty} m(u_n + b) - m(v_n + b) = \lim_{n \to \infty} m(u_n - v_n) = m \times 0 = 0.$$ 

  - The following theorem illustrates that any continuous function whose domain is a closed bounded interval is uniformly continuous:
Theorem 3.17: A continuous function $f : [a, b] \rightarrow \mathbb{R}$ is uniformly continuous.

- **Proof:** We will argue by contradiction. If it is not uniformly continuous, that means that there are sequences $\{u_n\}$ and $\{v_n\}$ in $[a, b]$ such that $\lim u_n - v_n = 0$ but $\lim f(u_n) - f(v_n) \neq 0$. Since the second limit is not zero, there must be some number $\epsilon > 0$ so that there is no number $N$ such that $|f(u_n) - f(v_n)| < \epsilon$ for all $n > N$. But then, there must be infinite subsequences $\{u_{n_k}\}$ and $\{v_{n_k}\}$ so that $|f(u_{n_k}) - f(v_{n_k})| > \epsilon$ for every $k$.

Now comes the key step. We use the sequential compactness theorem to get subsequences of $u_{n_k}$ and $v_{n_k}$ which each converge to some point in $[a, b]$. In fact, since $\lim u_n - v_n = 0$, they must converge to the same point $x_0 \in [a, b]$. According to continuity of $f$, the value of $f$ evaluated on each of these sequences would converge to $f(x_0)$...but according to the previous paragraph the distance between them is always greater than $\epsilon$. This is a contradiction, and so our assumption that $f$ was not uniformly continuous must have been wrong.

**Homework**

**Read:** Read Section 3.4 on "Uniform Continuity.

**Do:** Please complete and submit problems 1, 3 and 6 from this section. (Note that 6 can be achieved by giving a counter-example, and one can be found by looking at my examples from this handout.)