2.1 Sequences and Convergence

- **Definition:** A sequence is a function $f$ whose domain is $\mathbb{N}$ and whose range is contained in $\mathbb{R}$. (A mathematician would write “$f : \mathbb{N} \to \mathbb{R}$.”) We often also write it in the form $\{a_1, a_2, \ldots\}$ where $a_n = f(n)$, but this obscures the key fact that the sequence is an example of a function.

- **Basic Example:** Consider the sequence $f(n) = (-1)^n \frac{n}{n+1}$. Using the notation more familiar to you from other classes, this would be $\{-1/2, 2/3, -3/4, \ldots\}$. Of course, the formula for $f$ is more explicit than just listing the initial terms.

- **Recursive Example:** Some sequences are defined recursively, where the value of each term is described in terms of the preceding terms. (In this case, the values of some initial terms must be specified separately.) Consider, the sequence

$$a_{n+1} = \begin{cases} a_n + 1/n & \text{if } a_n^2 \leq 2 \\ a_n - 1/n & \text{if } a_n^2 > 2 \end{cases}$$

where $a_1 = 1$. What are the first four terms in this sequence?

- **Formula-Free Example:** Sometimes, it is possible to describe the function $f$ even when an exact formula is not possible. For example, let $f(n)$ be the largest natural number less than or equal to $\sqrt{n^3}$. This sequence looks like $\{1, 2, 5, 8, 11, \ldots\}$ but I think the description above may be the best way to characterize it.

- **Deep Example:** Example 2.3 says that any natural number $n$ can be written in a unique way as $n = j(j+1)/2 + k$ with $1 \leq k \leq j + 1$ and we define $a_n = k/(j + 1)$. This sequence lists all of the rational numbers between 0 and 1 (excluding zero but including 1)! In a sense, this means that the set $\mathbb{Q} \cap (0, 1]$ has the same size as the set $\mathbb{N}$. (Cantor’s Notion: two sets have the same size if their elements can be paired up somehow.) Perhaps this makes sense since both $\mathbb{N}$ and $\mathbb{Q} \cap (0, 1]$ are infinite, so you would want to say that two infinite sets have the same size. However, Cantor also proved that the set $\mathbb{R} \cap (0, 1]$ is not the same size, it is larger. This is one of those odd consequences of the Completeness Axiom that makes some people uncomfortable with it: infinity now comes in different sizes! (However, that is beyond the scope of this class. I mention it only for your interest.)

- **Infinite Series:** If $\{a_n\}$ is a sequence then the sequence $\{s_n\}$ with

$$s_n = \sum_{k=1}^{n} a_k$$

is an infinite series and $s_n$ is called an $n^{th}$ partial sum.
**Definition of Convergence:** A sequence \( \{a_n\} \) is said to converge to the number \( a \) provided that for every positive number \( \epsilon \) there is a natural number \( N \) such that
\[
|a_n - a| < \epsilon \quad \text{for all} \quad n \geq N.
\]
This can be expressed conveniently in the notation
\[
\lim_{n \to \infty} a_n = a.
\]

- Of course, we would like to think of this as meaning “the values of the sequence \( a_n \) get closer and closer to \( a \) as \( n \) gets larger”, but that description is vague and misleading. (For instance, it does not have to be the case that \( a_{2010} \) is closer to \( a \) than \( a_{2009} \) for the sequence to converge to \( a \)!) Whatever we may want it to mean, the definition must be interpreted literally. Convergence means no more and no less than what the definition says it means: **eventually the sequence stays between \( a + \epsilon \) and \( a - \epsilon \) no matter how small \( \epsilon \) may be.**

- **Theorem:** A sequence either does not converge to any number or it converges to one number \( a \). (It is not possible for there to be a sequence that converges to two different values at the same time.)

- **Proof:** Suppose that a sequence \( \{a_n\} \) converges to \( a \) and \( a' \) with \( a < a' \). Take the numbers \( N \) and \( N' \) which the definition of convergence gives for \( a \) and \( a' \) respectively when \( \epsilon = (a' - a)/4 \). By the Archimedean Principle, there is a number \( n \) which is bigger than both \( N \) and \( N' \). For this \( n \), it must be the case that
\[
a - \epsilon < a_n < a + \epsilon \quad \text{and} \quad a' - \epsilon < a_n < a' + \epsilon.
\]
But, there is a problem, Compare the upper limit on the first inequality with the lower limit on the other:
\[
a + \epsilon = \frac{a}{2} + \frac{a' + a}{4} < \frac{a'}{2} + \frac{a' + a}{4} = a' - \epsilon.
\]
We can prove using Positivity Axiom that this is a contradiction since \( a_n \) is supposed to be less than the smaller one and bigger than the larger one at the same time. \( \blacksquare \)

- **How to prove something converges:** The basic technique is to write out the inequality from the definition, and then manipulate it until it is in a form where the Archimedean Principle applies.

- **Example:** Prove that \( \lim_{n \to \infty} \frac{n}{n + 1} = 1 \).

  We would need to find an \( N \) such that \( n \geq N \) implies \( \left| \frac{n}{n+1} - 1 \right| < \epsilon \). Now, we use a ‘reduction process’ to determine another inequality which would (read in reverse!) imply the one we need:
\[
\left| \frac{n}{n+1} - 1 \right| < \epsilon \quad \iff \quad \left| \frac{1}{n+1} \right| < \epsilon \quad \text{(just algebraic simplification)}
\]
\[
\begin{align*}
\frac{1}{n + 1} &< \epsilon \quad \text{(because \(n\) is positive)} \\
\frac{1}{\epsilon} &< n + 1 \quad \text{(cross multiplying)} \\
\frac{1}{\epsilon} - 1 &< n \quad \text{(just add one to each side)}
\end{align*}
\]

Aha! we can find an \(n > \frac{1}{\epsilon} - 1\) by the Archimedean Principle! Then, following the steps in reverse would give us what we need.

**Another Example (done in the book):** The book similarly illustrates that \(\lim_{n \to \infty} \frac{1}{n} = 0\).

(Read the proof there and then we can use this useful fact frequently.)

- Sometimes, it is difficult to prove convergence directly and so an *indirect* method (via another sequence) is needed.

**Lemma 2.9 (page 28): Comparison Lemma:** If \(\lim_{n \to \infty} a_n = a\) and there exist numbers \(C \geq 0\) and \(N_1 \in \mathbb{N}\) such that

\[
|b_n - b| \leq C|a_n - a|
\]

then the sequence \(b_n\) converges to \(b\).

**Proof:** We need to find an \(N\) so that \(|b_n - b| < \epsilon\) when \(n > N\). Using the reduction process and assuming that \(n\) is bigger than the \(N_1\) from the statement of the lemma, we get

\[
|b_n - b| < k \\
\Leftrightarrow C|a_n - a| < \epsilon \quad \text{(because if \(n > N_1\) then this works)} \\
\Leftrightarrow |a_n - a| < \epsilon/C \quad \text{(just multiply by \(C\)).}
\]

But, since \(\{a_n\}\) converges to \(a\) there is some number (which we’ll call \(N_0\)) such that this last inequality is true for every \(n > N_0\). Now, we let \(N = \max(N_0, N_1)\) be the bigger of the two indices and each step of the reduction process works.

**Example:** Prove that \(\lim_{n \to \infty} \frac{5n^2 + 2n - 1}{n^3 + 3n^2 + 3} = 0\). We will apply the comparison principle: Since making the numerator bigger and the denominator smaller increases the value, and since \(n^2 < n^3/6\) when \(n > 6\), we know

\[
\left| \frac{5n^2 + 2n - 1}{n^3 - 3n^2 + 3} - 0 \right| \leq \left| \frac{5n^2 + 2n^2}{n^3 - 3(n^3/6)} \right| = 14 \left| \frac{1}{n} - 0 \right| \text{ if } n \geq 6.
\]

Now we can apply the Comparison Lemma with \(C = 14\) and \(N_1 = 6\) using the established fact that \(\{1/n\}\) converges to 0.

**Homework:** Read pages 23–28 and do 2.1: 1, 2 (no comparison lemma), 7