Math 311 Handout: August 27, 2010

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Preliminaries Reviewed

- Try to forget what you already know about numbers. We are going to work with certain explicit assumptions (axioms) and see what follows from those. (The idea: Either "we don't really know what numbers are" or "we want to make the assumptions that underlie them explicit".)

- The Field Axioms on page 3 set up the basic properties of addition and multiplication. (Among the special things we obtain here are the special numbers 0 and 1, the additive inverse $-a$ for any number a, and the mulitplicative inverse a^{-1} for any non-zero number. We use this to define $a - b$ which just means $a + (-b)$ and a/b which just means ab^{-1}.)

Question 1: Prove the Distributive Law for Differences: $c(b - a) = cb - ca$

Answer: (by Alex and Rob)

$$\begin{aligned} c(b-a) &= c(b+(-a)) \text{ (by definition of } b-a\text{)} \\ &= cb + c(-a) \text{ (by distributive axiom for addition)} \\ &= cb + c(-1)(a) \text{ (using } -a = (-1)a \text{ which we proved last time)} \\ &= cb + (-1)(ca) \text{ (using commutativity and associativity of multiplication)} \\ &= cb + (-ca) = cb - ca. \end{aligned}$$

$\square$

Note: Now that we've proved this here, we can use this fact (that multiplication distributes over differences) without having to reprove it. **Our toolbox is starting out small, but will grow as the semester progresses!**

- The Positivity Axioms on page 3 divide the numbers into those that are positive (called $\mathcal{P}$) and those that are not.
(We use this to define $a < b$ which just means "$b - a$ is positive".)

Question 2: Prove that $a < b$ and $b < c$ implies that $a < c$.

Answer: By definition, $x < y$ means $y - x$ is a positive number. So, we know from the statement that $b - a$ and $c - b$ are positive. The Positivity Axiom tells us that the sum of positive numbers is positive, so $(b - a) + (c - b) = c - a$ is positive...which precisely means that $a < c$! $\square$

- ***Important:*** We are just *one* axiom away from the real numbers. What we have said so far does not give us enough to say that there is a number $\sqrt{2}$ or discover calculus. The next section introduces this last axiom, and that's what really gives calculus its power.

1.1 The Completeness Axiom

- A set $S \subseteq \mathbb{R}$ is said to be "inductive" (see page 5) if it contains 1 and also contains $x+1$ whenever it contains x. The smallest inductive set is $\mathbb{N}$. This is technically the definition of $\mathbb{N}$, the natural numbers, though you should probably think of it as $\mathbb{N} = \{1, 2, 3, \ldots\}$. The set of integers, $\mathbb{Z}$, is defined as the union of $\mathbb{N}$, with $\{0\}$ and the additive inverses of the elements of $\mathbb{N}$. Finally, the rational numbers, $\mathbb{Q}$, is the subset of $\mathbb{R}$ whose elements are of the form p/q with p chosen from $\mathbb{Z}$ and q chosen from $\mathbb{Q}$. (The definition of "inductive" is also used to introduce the method of proof by induction, but since that is something you learned in Math 295 I won't spend much time on it.)
- The axioms listed so far do *not* characterize the real numbers. Note, for example, that the set $\mathbb{Q}$ of *rational* numbers satisfy all of the axioms given so far. This is certainly not enough to do calculus since we want to show, for example, that if $a < b$ and $f(a) < 0 < f(b)$ then $f(c) = 0$ for some $a < c < b$ for a continuous function f. This is *not* true if we consider only rational numbers. (For instance, $f(x) = x^2 - 2$, $a = 0$ and $b = 2$.)
- We need to add one more axiom, and to do so we need one more definition:

  Definition: A nonempty set S of real numbers is said to be *bounded above* (bounded below) provided that there is a number c, called an upper (lower) bound, such that $x \leq c$ ($x \geq c$) for every x in S.

The Completeness Axiom: Suppose that S is a nonempty set of real numbers that is bounded above. Then, among the set of upper bounds for S there is a smallest, or least, upper bound which we call $\sup S$.

Question 1: Prove that it follows from the other axioms that if S is bounded below then there is a greatest lower bound, $\inf S$.

- With just this additional axiom, we now have the whole real number system. In particular, this necessarily adds the irrational numbers since (for example) the set $S = \{x \text{ in } \mathbb{R} : x^2 < 2\}$ is bounded above and $\sup S$ is a number whose square is 2. (Both of these statements need to be proved from the axioms, but I'm hoping you'll just accept them for the moment.) As you probably learned before, no rational number has 2 as its square, so this number is necessarily *irrational*.
- In the next section, we will learn more about how the rational and irrational numbers are distributed among the set of all real numbers.
- Are these axioms for the real numbers "the right ones"? Perhaps, or perhaps not. They have worked very well (we discovered radio waves using them and cell phones seem to work, right?), but there are some strange consequences that might give you reason to question. Take a look at the Banach-Tarski paradox or the well-ordering property, which make it seem as if something is really wrong. If I have some extra time, I may show you that the notion of an infinitely large set has already gotten more complicated just from adding the completeness axiom. In particular, although the set of positive integers, the sets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are all infinite sets of the "same size", the set $\mathbb{R}$ is *larger* than those! (Cantor's Diagonal proof.)

➢ **NOTE ON LOGIC:** Writing proofs will require you to use logic. It may be that you have not yet had a formal class in logic, but I think that you already *know* it. You just need practice thinking about how to apply it. It may be easier to think about some more concrete examples. For instance, consider the following statements:

1. All Americans are multi-billionaires.
2. Some Americans are multi-billionaires.
3. No Americans are multi-billionaires.

If we wanted to prove or disprove some of these statements, what could we do with two examples? I mean, using *only* the fact that Bill Gates and Warren Buffett *are* two American multi-billionaires, what can we conclude?

Well, just that example *disproves* (3), right? From knowing that there are two, we can say for certain that the third statement is false. Similarly, it *proves* that (2) is true...*some* American's are multi-billionaires. However, it tells us nothing about (1). It neither proves nor disproves that *all* Americans are multi-billionaires simply to know about Gates and Buffett. We would have to either show that every other American is also a multi-billionaire to prove that it is true *or* just give one example of someone who is not to show that it is false.

Homework: Read the section entitled "Preliminaries" and Section 1.1. Do problems 1.1 # 15, 19 (Prove rigorously from the axioms without using what you've learned about numbers elsewhere.)

This homework is due in class on Wednesday.