Key Ideas: The Big Picture

There is a rather general way to consider \( \text{div}, \text{grad} \) and \( \text{curl} \) as all being examples of a single operator "\( d \)" and then to consider most of our major theorems in this class as being special cases of a single fact called the \textit{Generalized Stokes' Theorem}. I will try to explain this further in this note for those who are interested, but please be aware that this really goes beyond what you are expected to know in this class.

The General form of "FTC": We've seen a bunch of examples of theorems of a common type. In fact, when I learned this material we learned them all at once as something called the "Generalized Stokes Theorem" that includes the Divergence theorem, Green's theorem and the usual Fundamental Theorem of Calculus as special cases. The general form is that the integral of some differential form "\( df \)" over an \( n \)-dimensional thing\(^1\) \( \mathbb{R} \) is the same as the integral of just \( f \) over the boundary \( \partial R \) (which is an \((n - 1)\)-dimensional thing):

\[
\int_R df = \int_{\partial R} f
\]

It looks like the \( d \) just slid down a bit and became a \( \partial \)! Of course, this \( d \) is a bit different than anything you've seen before. Below, I will describe it in more detail for those who may be interested. However, you should note that it is really not part of this class and should be considered as extra-curricular.

In any case, the amazing thing about this equality is that it actually relates two very different sorts of operators. The one on the left is an actual derivative in the sense of calculus (whether it is gradient or curl or divergence or whatever) while the one on the right is just the "boundary" operator which is purely geometric! In other words, the beauty of the FTC (not at all apparent when you first saw it in Calc 1) is that it gives a deep relationship between objects of calculus (differentiation and integration) and objects of geometry (boundary and dimension).

The Wedge Space of Differential Forms

The objects \( dx, dy \) and \( dz \) that you've seen floating around in calculus are called \textit{differentials}. (I generally don't say much about them in this course, even though they are important, simply because I have to cut \textit{some} things in order to make room for all of the material. I had thought that this would only affect math majors, who would probably encounter these objects in another class. However, last year it was brought to my attention that students in chemistry are expected to know something about differentials as well!) We use them to make \textit{differential forms}. The differential forms are made by "wedging" together some of these differentials, multiplying them by functions

\(^1\) \( R \) is actually called a \textit{manifold}. You can think of it as being an \( n \)-dimensional space, but unlike the spaces \( \mathbb{R}^3, \mathbb{R}^2 \) and \( \mathbb{R}^3 \) that we've considered in this class, \( R \) may not be \textit{flat}. It has a geometry and a topology all its own. As we learned from General Relativity, the same is true of the universe that we live in. It is a 4-dimensional manifold and although we do not know its overall shape yet (there are guesses), we do know that it is not locally flat!
and adding them together. We usually like there to be the same number of wedges in each term of the sum and we call that number the degree of the form.

So, for instance,

\[ P(x, y, z) \, dx + Q(x, y, z) \, dy + R(x, y, z) \, dz \]

is an example of a 1-form. Similarly,

\[ P(x, y, z) \, dy \wedge dz + Q(x, y, z) \, dz \wedge dx + R(x, y, z) \, dx \wedge dy \]

is what we would call a 2-form. Actually, just plain functions like \( f(x, y, z) \) are zero forms and then there are also 3-forms that look like \( f(x, y, z) \, dx \wedge dy \wedge dz \).

There is only one rule you need to know about the wedge product \(^2\). It is that \( a \wedge b = -b \wedge a \). From this you can conclude that \( a \wedge a = 0 \) (which is why you don’t see any term of the form \( dx \wedge dx \) in the 2-form above). You can also use it to order the differentials in some simple way, which is why you don’t see anything of the form \( dy \wedge dx \).

We denote by \( \bigwedge^k \) the space of all \( k \)-forms. Note that \( \bigwedge^0 \) is just the space of ordinary functions, \( \bigwedge^1 \) can be interpreted as being the set of all vector fields if we let \( i = dx, j = dy \) and \( k = dz \), \( \bigwedge^2 \) and \( \bigwedge^3 \) again are vector fields (now with \( i = dy \wedge dz \) and functions respectively! (This, however, is only because we are in three dimensional space. If the dimension of the underlying space was larger then there would be more of these wedge spaces with more complicated dimension counts.) In general, the space \( \bigwedge^k \) has dimension \( \frac{n!}{(n-k)k!} \) if the underlying space is \( n \) dimensional.

### Taking Derivatives of Differential Forms

The total derivative \( d \) is a single operator that takes any \( k \)-form \( \omega \in \bigwedge^k \) to a \((k+1)\)-form \( d\omega \in \bigwedge^{k+1} \). In Calculus 1 you learned just a bit about taking derivatives of these differentials. In particular, you learned that if \( u = f(x) \) then \( du = f'(x) \, dx \) in order to do some \( u \)-substitutions. There is a more general form of this in higher dimensions.

The total derivative operator \( d \) takes a function \( u = f(x, y, z) \) to \( du = f_x \, dx + f_y \, dy + f_z \, dz \). More generally, if \( \omega = f \omega \) where \( f \) is a function and \( \omega \) is any wedge of differentials, then \( d\omega = (df) \wedge \omega \).

So, if we start with a 0-form \( \omega = f(x, y, z) \in \bigwedge^0 \) we see

\[ d\omega = f_x \, dx + f_y \, dy + f_z \, dz \]

Note that this is the gradient vector of \( f \).

What if we start with a 1-form \( \omega = P \, dx + Q \, dy + R \, dz \)? Then

\[ d\omega = (P_x \, dx + P_y \, dy + P_z \, dz) \wedge dx \]

\(^2\)If you want to know more about the wedge than just this, you can check out a paper I wrote with two C of C undergrads. The wedge product turns out to be very important in my research on waves and particles. Search the Web for “pedings reisal kasman” to see the paper which was published in the Proceedings of the American Mathematical Society and has attracted attention from both algebraic geometers and particle physicists!
\[(Q_x dx + Q_y dy + Q_z dz) \wedge dy
\]  
\[+(R_x dx + R_y dy + R_z dz) \wedge dz
\]  
\[=(R_y - Q_z) dy \wedge dx + (P_z - R_x) dz \wedge dx + (Q_x - P_y) dx \wedge dy.
\]

Note that this is the curl!

And if we start with a general 2-form \( \omega = P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy \) then

\[d\omega = (P_x dx + P_y dy + P_z dz) \wedge dy \wedge dz
\]  
\[+(Q_x dx + Q_y dy + Q_z dz) \wedge dz \wedge dx
\]  
\[+(R_x dx + R_y dy + R_z dz) \wedge dx \wedge dy
\]  
\[=(P_x + Q_y + R_z) dx \wedge dy \wedge dz,
\]

the divergence!

Now, it can be shown that no matter what \( \omega \) is, \( d^2 \omega = 0 \). (So, doing \( d \) and then \( d \) again “kills” anything.) This captures (and generalizes) the “div kills curl” and “curl kills grad” facts that otherwise seem separate. More interestingly, the extent to which \( d\omega \) is zero even if \( \omega \) is not already \( d \) of something else reveals a lot of information about the topology of the space (this is the start of a subject called “cohomology”)! Here is what Wikipedia has to say in its “de Rham Cohomology” entry:

Forms which are the image of other forms under the exterior derivative are called exact and forms whose exterior derivative is 0 are called closed; the relationship \( d^2 = 0 \) then says that exact forms are closed. The converse, however, is not in general true; closed forms need not be exact. A simple but significant case is the 1-form of angle measure on the unit circle, written conventionally as \( d\theta \). There is no actual function \( \theta \) defined on the whole circle for which this is true; the increment of \( 2\pi \) in going once round the circle in the positive direction means that we can’t take a single-valued \( \theta \). We can though by removing just one point, changing the topology. The idea of de Rham cohomology is to classify the different types of closed forms on a manifold \( M \). One performs this classification by saying that two closed forms \( \alpha \) and \( \beta \) in \( \bigwedge^k(M) \) are cohomologous if they differ by an exact form, that is, if \( \alpha - \beta \) is exact. This classification induces an equivalence relation on the space of closed forms in \( \bigwedge^k(M) \). One then defines the \( k \)-th de Rham cohomology group to be the set of equivalence classes, that is, the set of closed forms in \( \bigwedge^k(M) \) modulo the exact forms.

The algebra of these groups gives you a clue as to the topology of the manifold \( M \). We saw a hint of this when we realized that a vector field could be conservative in some simply connected region but not conservative in a region with holes. If \( M = \mathbb{R}^2 \) is the plane then every closed form is exact. But if \( M = \mathbb{R}^2 - \{(0,0)\} \) is the plane with the origin missing then there are some closed forms that are not exact. Their existence (and we can “see” them in the cohomology group) indicates that there is a hole.

I’m not sure if that will make any sense to you. If it does, great. If not, don’t worry about it. But, if you’d like more explanation, feel free to ask!