Key Ideas: The Divergence Theorem

- The point of today’s lecture is to learn just one more theorem relating different kinds of integrals. As in the other versions of the Fundamental Theorem of Calculus, it will change the number of dimensions of the object over which we’re integrating. In particular, it turns a triple integral into a surface integral (or vice versa).

**The Divergence Theorem (also known as “Gauss’ Theorem”):** Let \( E \) be a solid region in space and let \( S = \partial E \) be the boundary of \( E \) with the outward orientation. Then as long as \( \mathbf{F}(x, y, z) \) has continuous partial derivatives around \( E \) it is true that

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div} \mathbf{F} \, dV.
\]

- This theorem is useful because it is often easier to evaluate a triple integral of a scalar function than to evaluate a surface integral of a vector field. Don’t you agree? There are no dot products; no parametrization is necessary, etc!

- **Example 1 p. 1101:** Find the flux of \( \mathbf{F} = \langle z, y, x \rangle \) over the unit sphere \( x^2 + y^2 + z^2 = 1 \).
  (Hint: A sphere of radius \( r \) has volume \( \frac{4}{3} \pi r^3 \).)

- **Example 2 p. 1101:** Find the flux of \( \mathbf{F} = \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle \) over the surface \( S \) that is the boundary of the region \( E \) bounded by the parabolic cylinder \( z = 1 - x^2 \) and the planes \( z = 0, y = 0, \) and \( y + z = 2 \). (This surface would be a pain to parametrize, but the triple integral is rather easy.)

- Section 16.9 also spends some time now discussing something I mentioned to you earlier: the interpretation of \( \text{div} \mathbf{F} \) as the amount of expansion or compression at a point. In particular, if there are more (or bigger) vectors pointing towards a point than away from it, the divergence there will be negative; we call such a point a sink. If more is going away from it than towards it, the divergence would be positive; that’s called a source. And, if there is just as much flowing away from it as towards it then the divergence will be zero there.

- Now, you can see why the divergence theorem is true. If you add up all of the “sources” and “sinks” within a closed surface, you’ll find out how much flow there is leaving from inside there. In other words the integral on the right (adding up the divergences) will give you exactly the flux integral on the left.
This gives us another way to understand the fact that if \( \mathbf{G} = \text{curl} \, \mathbf{F} \) then \( \int_S \mathbf{G} \cdot d\mathbf{S} \) over a closed surface \( S \) is equal to zero. We have three so far: (a) use Stokes’ theorem and note that \( S \) has no boundary; (b) cut \( S \) into two pieces that have the same boundary (but with opposite orientations) and so the integral over the two pieces differ by a sign and equal zero when added together or (c) use the divergence theorem but note that \( \text{div} \, \mathbf{G} = \text{div} \, \text{curl} \, \mathbf{F} = 0 \).

Students sometimes expect there to be two versions of this theorem: one analogous to Green’s Theorem and then a more general one like Stokes’ Theorem. They would be absolutely correct, but we will not be seeing both of them in this course. This version we have learned today is analogous to Green’s Theorem. Note, for example, that in both this theorem and Green’s theorem there is only one region with a given boundary while in Stokes’ Theorem there are many possible surfaces with a given closed curve as its boundary. To see the generalization of this theorem matching Stokes’ Theorem, we would have to work in four dimensions where there are many regions having a given closed surface as its boundary. Those of you in physics know that 4-dimensional space is not a silly idea... modern physics requires it! To see the general case, the one that works in any number of dimensions, come to class on Friday when I’ll go a bit beyond the material in the syllabus.

**Idea of the Proof of the Divergence Theorem:** Let \( \mathbf{F}(x, y, z) = \langle P, Q, R \rangle \) be a vector field and \( B = [a, b] \times [c, d] \times [j, k] \) be the box shaped region in space with \( a \leq x \leq b, c \leq y \leq d \) and \( j \leq z \leq k \). The surface \( S = \partial B \) is made up of six rectangles. Let us give names to the two of them that are parallel to the \( yz \)-plane: call \( S_1 \) the side of \( B \) in the plane \( z = j \) and \( S_2 \) the side in \( z = k \). We can parametrize these as \( \mathbf{r}(x, y) = \langle x, y, j \rangle \) and \( \mathbf{r}(x, y) = \langle x, y, k \rangle \) for \( a \leq x \leq b \) and \( c \leq y \leq d \)...but we have to remember that the orientation on \( S_1 \) points down \( x \)-direction while the orientation on \( S_2 \) points in \( up \) so that

\[
\int_{S_1} \mathbf{F} \cdot d\mathbf{S} + \int_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_a^b \int_c^d -R(x, y, j) \, dy \, dx + \int_a^b \int_c^d R(x, y, k) \, dy \, dx
\]

\[
= \int_a^b \int_c^d \left(R(x, y, k) - R(x, y, j)\right) \, dy \, dx.
\]

Using the old FTC from Calc I we know that \( R(x, y, k) - R(x, y, j) = \int_j^k R_z(x, y, z) \, dz \) so the integral above becomes

\[
\int_{S_1} \mathbf{F} \cdot d\mathbf{S} + \int_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_a^b \int_c^d \left(\int_j^k R_z \, dz\right) \, dy \, dx = \iiint_B R_z \, dV.
\]

The same argument gives us that the integrals on the planes parallel to the \( yz \)-plane add up to \( \iiint_B P_x \, dV \) and the integrals on the planes parallel to the \( xz \)-plane add up to \( \iiint_B Q_y \, dV \). Thus, the entire integral adds up to

\[
\int_S \mathbf{F} \cdot d\mathbf{S} = \iiint_D (P_x + Q_y + R_z) \, dV = \iiint_B \text{div} \, \mathbf{F} \, dV.
\]
Homework

Section 16.9: Read and do 1, 5–15, 19–20

Wrap-up / Answering your questions: On Monday you will be able to ask me questions about the sample problems and the final exam.

“Through the Looking Class” Friday: On Friday I will explain how mathematicians have made a simplified and beautiful theory out of the messy collection of theorems and definitions that we learned in this class...though that does take us a bit beyond the syllabus and will not appear on any graded assignments.

Don’t miss the final!: The final itself is in our usual classroom on Saturday December 10th at 8AM.