Key Ideas: More on Stokes’ Theorem

- If \( S \) is a closed surface then \( \partial S \) is just the empty set. Note then that we can conclude that the flux of \( F = \text{curl} \ G \) is zero on a closed surface! (This somehow “rhymes” with the fact that the integral of \( F = \nabla f \) is zero on a closed path. To see a hint of why, be sure to attend the last day of class.)

- Note that Stokes’ theorem reduces to Green’s theorem if the surface \( S \) happens to lie in the \( xy \)-plane. So, it is not just analogous to but really a generalization of Green’s theorem. The interesting analogy is between Stokes’ Theorem and the FTC for path integrals.

- Compare Stokes’ Theorem to the FTC for path integrals of vector fields:

  If \( F \) happens to be the gradient vector field of some function \( f \) (in other words, if \( F = \nabla f \)) then

  \[
  \int_C F \cdot dr = f(b) - f(a)
  \]

  Thus, to evaluate the integral on the left we can “anti-differentiate” \( F \) to get \( f \) and just look at its values at the boundary. A consequence is that the integral of \( F \) around a closed curve is zero (if \( F \) is conservative)!

  If \( F \) happens to be the curl of some vector field \( G \) (in other words, if \( F = \text{curl} \ G \)) then

  \[
  \int_S F \cdot dS = \int_{\partial S} G \cdot dr
  \]

  Thus, to evaluate the integral on the left we can “anti-differentiate” \( F \) to get \( G \) integrate it over the boundary. A consequence is that the flux of \( F \) over a closed surface is zero (if \( F = \text{curl} \ G \)).

- The orientation for a point is just the assignment of either “+” or “−” to the point (two choices, as usual). One thing we can notice in this analogy is that if \( C \) is an oriented curve, \( \partial C \) should be the endpoints \( a \) and \( b \). The fact that you evaluate at these endpoints, subtracting one and adding the other, has to do with the induced orientation (just like in Stokes’ theorem). So, if you ever evaluated an old fashioned integral with the FTC but switched the order on the subtraction by mistake, you were really just getting the orientation wrong!

**Question 1:** Example 1 on page 1095: Evaluate the integral of \( F = \langle -y^2, x, z^2 \rangle \) on the curve \( C' \) that is the intersection of \( y + z = 2 \) and \( x^2 + y^2 = 1 \) oriented counterclockwise from above.

**Question 2:** Let \( F \) be a “nice” vector field (defined everywhere in \( \mathbb{R}^3 \) with continuous partial derivatives). Which of these four things must be equal?

1. The surface integral

   \[
   \iint_{S_1} \text{curl} F \cdot dS
   \]
where $S_1$ is the top half of the unit sphere $x^2 + y^2 + z^2 = 1$ with outward orientation.

2. The surface integral

$$\iint_{S_2} \text{curl} \mathbf{F} \cdot d\mathbf{S}$$

where $S_2$ is the graph of $z = x^2 + y^2 - 1$ for $(x, y)$ in the unit disk $\{(x, y): x^2 + y^2 \leq 1\}$ with the standard orientation for graphs.

3. The line integral

$$\oint_{C} \mathbf{F} \cdot d\mathbf{r}$$

where $C$ is the circle parametrized by $\mathbf{r}(t) = \langle \cos(2t), \sin(2t), 0 \rangle$ for $0 \leq t \leq \pi$.

4. The line integral

$$\oint_{C} \mathbf{F} \cdot d\mathbf{r}$$

where $C$ is the circle parametrized by $\mathbf{r}(t) = \langle \cos(-t), \sin(-t), 0 \rangle$ for $0 \leq t \leq 2\pi$.

**Section 16.8:** 15–16. If you want more, you can also do 32 and 33 on *page 1108*. 