Key Ideas: Surface Integrals and Flux (continued)

- Let’s just work through some examples of flux integrals now before moving on to our next big theorem.

**Question 1:** Consider the surface \( \mathbf{r}(u, v) = (\sin u, \cos u, uv + u) \) for \( 0 \leq u \leq 3\pi \) and \( 0 \leq v \leq 1 \). What is the flux of the vector field \( \mathbf{F}(x, y, z) = (xz, yz, \tan(xy) + e^{z^3}) \) over this surface? (Note: The surface has the orientation induced by the parametrization.)

This integral will become

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{3\pi} \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv.
\]

We need to find \( \mathbf{F}(\mathbf{r}(u, v)) \) and \( \mathbf{r}_u \times \mathbf{r}_v \) and take their dot product:

\[
\mathbf{F}(\mathbf{r}(u, v)) = \left( (uv + u) \sin u, (uv + u) \cos u, \tan(\sin u \cos u) + e^{(uv+u)^3} \right)
\]

and

\[
\mathbf{r}_u \times \mathbf{r}_v = (-u \sin u, -u \cos u, 0).
\]

So their dot product is

\[-u(uv + u) \sin^2 u - u(uv + u) \cos^2 u = -u^2v - u^2.\]

Thus the integral becomes

\[-\int_0^1 \int_0^{3\pi} u^2v + u^2 \, du \, dv = -\frac{27\pi^3}{2}.\]

Now, what does that mean? To figure that out, we need to work out what the vector field looks like and what the orientation looks like.

The orientation is given by letting \( \mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \) at every point on the surface. But, we can get a pretty good idea just by looking at one point. If \( u = 0 \) and \( v = 0 \) then we’re looking at the point \((0, 1, 0)\). The normal vector \( \mathbf{n} \) there is \( (0, -1, 0) \)...the one pointing in towards the origin. This is the orientation that comes from the parametrization. So, the flux integral should be measuring how much of the flow is coming into the middle through this surface.

Now, looking at the vector field on the computer may help us visualize the flow. However, it turns out that the vertical component of the flow is so strong that we can’t see much else. (The flow is going up very quickly.) We can see both from the computation above and from the shape of the surface that this vertical flow has nothing to do with our answer, so we can just set it equal to zero. Then we see that the flow actually is going out through the surface, in the direction opposite to our orientation. That is why the final value is negative!
**Question 2:** (Example 4, p. 1089) Find the flux of \( \mathbf{F} = \langle y, x, z \rangle \) over the surface of the solid bounded by \( z = 1 - x^2 - y^2 \) and \( z = 0 \) with the usual (outward) orientation for closed surfaces.

- Take a look at it on the computer. Can you tell if you expect a positive or negative answer?
- We need to handle the top and bottom parts of the surface separately. Let us call \( S_1 \) the disk at the bottom and \( S_2 \) the parabolic dome.
- We can describe \( S_1 \) and its orientation pretty easily. First, we think of it as the graph of \( z = 0 \) for \( (x, y) \) in the unit disk \( D \). But, we don't want the usual orientation. The unit normal vector here should just be \( \langle 0, 0, -1 \rangle \) at every point, since it should be pointing outward from the surface (and because it is a horizontal plane).

Then

\[
\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_D \langle y, x, 0 \rangle \cdot \langle 0, 0, -1 \rangle \, dA = 0.
\]

There is no integrating to do here since we're just integrating zero over the disk. Why? Look at the bottom of the object along with the vector field and you'll see that there is nothing flowing in or out through that plane. This is because when \( z = 0 \) the vector field has no vertical component.

- Things are more interesting if we look at \( S_2 \). We think of it as part of the graph of \( z = 1 - x^2 - y^2 \). This time, we just use the regular choice of normal vector \( (\mathbf{n} = \langle -f_x, -f_y, 1 \rangle / \sqrt{1 + f_x^2 + f_y^2}) \) because this points up and that is the direction we want. Then

\[
\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_D \langle y, x, 1 - x^2 - y^2 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dA
\
= \iint_D 1 + 4xy - x^2 - y^2 \, dA
\
= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos \theta \sin \theta - r^2)r \, dr \, d\theta
\
= \pi/2.
\]

Finally, we add these two together to get the flux over \( S \):

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 0 + \frac{\pi}{2} = \frac{\pi}{2}.
\]

- What does this mean? It means that more flow is coming out through the surface than is going in. That may seem counter-intuitive to you, but that is because you are used to thinking that a moving fluid does not expand or contract (i.e. is divergence free). It is true that if the fluid is divergence free, then this can not happen. However, in this case it means that some sort of expansion must be going on inside this bullet shaped surface because more of the stuff is coming out than is going in.