Key Ideas: Surface Integrals and Flux

- **Orientation for a Surface:** Just as a curve can always be oriented in either of two ways, a surface (if it is orientable!) can be oriented in either of two ways. To specify an orientation for a surface, we need to identify one side of the surface as being "positive". For example, the standard orientation for a connected closed surface is that the outside is positive. The standard orientation for a graph is that up is positive. In general, however, to give an orientation for a surface you really need to specify it explicitly. However, even though these are the standard orientations, it is always possible to choose the other orientation.

- **Orientation by Normal Vectors:** At each point on a smooth surface \( S \) there are two unit normal vectors...one pointing in each direction. The most useful way to specify an orientation is to explicitly give a normal vector at each point that points out from the positive side. So, for instance, for a surface that is the graph of a function \( g \) we give the standard orientation by saying

\[
n = \frac{-g_x i - g_y j + k}{\sqrt{1 + g_x^2 + g_y^2}}
\]

since at any point \((x, y, g(x, y))\) on the surface \( n \) is the normal vector with positive \( z \)-component. Also, a parametrization gives an induced orientation by

\[
n = \frac{r_u \times r_v}{|r_u \times r_v|}.
\]

- **Warning:** There are surfaces which are not orientable. In today's group project, we will see that there are surfaces which cannot have one side colored red and another side colored blue...because they do not have two sides. The integrals we are talking about here only work for an orientable surface. Only orientable surfaces will be considered in this class from now on, but in theory before you consider doing anything we discuss below, you would first have to make sure that the surface is orientable before proceeding!

- **Definition:** We define the surface integral of \( F \) over \( S \) (or the flux of \( F \) across \( S \)) to be

\[
\iint_S F \cdot dS = \iint_S F \cdot n \, dS.
\]
What would this mean? Note that \( \mathbf{F} \cdot \mathbf{n} \) is a positive number when \( \mathbf{F} \) is pointing out of the surface in the positive direction, it is negative when \( \mathbf{F} \) is pointing towards the negative direction, and it is zero when \( \mathbf{F} \) is tangent to the surface. This gives us a way to think about what this number means. But how can we calculate it? It looks awfully messy because 

\[
\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.
\]

**GOOD NEWS:** At first it looks as if this might make things rather complicated. But the truth is that things get quite simple in the cases we do most often. In particular, remember that when we integrate \( dS \) we have to put in the term \( |\mathbf{r}_u \times \mathbf{r}_v| \) which is the length of the normal vector. Well, when you multiply \( \mathbf{n} \) by this you just get \( \mathbf{r}_u \times \mathbf{r}_v \) itself! In other words, the formula above simplifies as follows:

If \( S \) happens to be the graph \( z = g(x, y) \) over \( D \) then this becomes

\[
\int\int_D \mathbf{F}(x, y, g(x, y)) \cdot (-g_x, -g_y, 1) \, dA = \int\int_D (-Pg_x - Qg_y + R) \, dA
\]

and if \( S \) is a standard parametrized surface it becomes

\[
\int\int_D \mathbf{F}(x(u, v), y(u, v), z(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA.
\]

**Physical Interpretation:** If we interpret \( \mathbf{F} \) as a velocity field, the flux is the total amount of flow across the surface \( S \) in the direction specified by the orientation. For example, let us take \( S \) to be the unit disk with the upwards orientation. A fluid flow that is horizontal would have a flux of 0 because none of it is through the disk. A fluid flow that is downwards would have a negative flux while an upwards fluid flow would have a positive flux. A flow that was downwards on half the disk and symmetrically upwards on the other half of the disk would have a flux of zero as well because the net flow across the disk would be zero.

**Question 1:** Find the flux of the vector field \( \mathbf{F} = (-y, x, 0) \) over the graph of \( z = x^2 + y^2 \) over the square \([-1, 1] \times [-1, 1]\).

**Question 2:** Evaluate the flux of the vector field \( \mathbf{F} = (0, 0, z) \) on the boundary of the solid region \( E \) enclosed by \( z = 1 - x^2 - y^2 \) and \( z = 0 \). Interpret this number and its significance to a flow whose velocity field is \( \mathbf{F} \). (Hint: Since this is a closed surface, the default orientation points outwards. Hint: Use the parametrization \( \mathbf{r} = (r \cos \theta, r \sin \theta, 1 - r^2) \) for the top.)

**Homework**

**Section 16.7:** Read the whole section and do: #19–30.