Key Ideas: Surface Integrals

- **Review: Line Integrals:** It may be useful for you to recall at this point what we know about line integrals. We can integrate a function along a curve with respect to arclength. To do that we have to parametrize the curve and convert the line integral to an ordinary integral with respect to $t$. We also learned how to integrate a vector field along a curve. Generally, doing this involves converting it into a line integral of a function and from there by parametrizing the curve to an ordinary integral with respect to $t$. However, in the special case that the vector field is conservative, we can avoid all of that by using the FTC. Also, if the curve is a closed curve in the plane, you can use Green’s Theorem to replace the line integral with a double integral. Remember also that for integrating vector fields along a curve the orientation of the curve was relevant. (Remember the difference between the wind being at your back as opposed to walking into the wind?)

- **Surface Integrals:** Today we will be learning about surface integrals. In many ways, what we see will be entirely analogous to the case of line integrals. In particular, today we will learn how to integrate a scalar function over a surface by parametrizing the surface, we will later learn how to integrate a vector field over an oriented surface by turning it into an integral of a scalar function, and eventually we will learn of a sort of FTC that will apply.

- **Definition of a Surface Integral:** Given a function $f(x, y, z)$ defined on a surface $S$ in space we can compute the integral of $f$ over the surface $S$. Of course, the definition of this integral is given in terms of a Riemann sum. In particular, you break up the surface along some gridlines and multiply the area of the “rectangular” region between the gridlines by the value of a function at some point in that region and add them up. Taking the limit as the maximum distance between the gridlines goes to zero will give you the integral.

- **One example to consider:** if $f(x, y, z)$ is a function that describes the density of something at the point $(x, y, z)$ (for example, the density of cells/sq.cm. for a bacterial population on the surface of a toilet bowl) then the surface integral gives the total amount (that is, the size of the population living) on $S$.

- **Computing a Surface Integral:** In some of the applications that I plan to show in class today, the value of a surface integral is estimated using the Riemann sum definition above. So, it is sometimes useful. However, in this class we will always be able to compute the integrals analytically using the following formula:

$$
\iint_S f(x, y, z) \, dS = \iint_D f(r(u, v))|r_u \times r_v| \, dA.
$$

The idea here is to parametrize $S$ by some vector function $r$ defined on a domain $D$ in the $uv$-plane and hence convert the surface integral into an ordinary double integral.
How to think of it: Note that the $|\mathbf{r}_u \times \mathbf{r}_v|$ term added to the integrand is the thing we had to integrate to get the area. You can think of this as a sort of “Jacobian” associate with the new coordinates $(u, v)$ that we’ve put on the surface. You can also think of it as an “area element” which is multiplied by the value of $f$ (which turns into an integral when summed).

**Question 1:** Write (but do not necessarily evaluate) an ordinary double integral whose value is the integral of the function $f(x, y, z) = x^2 + y^2$ on the surface of the sphere of radius 1.

- **Breaking Up:** If the surface can more naturally be viewed as the union of two or more surfaces then you can integrate over each of those separately and just add the results together.

**Question 2:** Let $S$ be the surface of the tetrahedron with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$. Find $\int_S xyz \, dS$.

- **Special Case: if $S$ is a graph:** As before, in the special case that $S$ is just the graph of some function, we can choose $\mathbf{r}(x, y) = (x, y, g(x, y))$. Then the formula above simplifies to:

$$\int_S f(x, y, z) \, dS = \int_D f(x, y, g(x, y)) \sqrt{g_x^2 + g_y^2 + 1} \, dA.$$  

**Question 3:** Evaluate the integral of the function $f(x, y, z) = \sqrt{5 - 4z}$ over the surface of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

- **Special Case: if $f(x, y, z) = 1$:** If you are integrating the constant function 1 over the surface $S$, then this just reproduces the formula from the previous section for the surface area of $S$

$$A(S) = \int_S 1 \, dS.$$  

**Homework**

**Section 16.7:** Don’t worry about reading the section yet since we are not really done with it. Do these problems: 5–18