Math 221 Handout: November 11, 2011

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**Question 1:** Let $C$ be the path in the plane that starts at $(0,0)$, goes straight up to $(0,1)$, straight across to $(2,1)$, down to $(2,0)$ and straight back to the origin. Let $\mathbf{F} = \langle y^2, x^2 \rangle$. Find the value of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ “the old fashioned way”. Then use Green’s Theorem to find the value using a double integral. Compare.

**Answer (without Green’s Theorem):** We can parametrize $C$ as

- $C_1: \mathbf{r}(t) = \langle 0, t \rangle \quad 0 \leq t \leq 1$
- $C_2: \mathbf{r}(t) = \langle t, 1 \rangle \quad 0 \leq t \leq 2$
- $C_3: \mathbf{r}(t) = \langle 2, 1-t \rangle \quad 0 \leq t \leq 1$
- $C_4: \mathbf{r}(t) = \langle 2-2t, 0 \rangle \quad 0 \leq t \leq 1$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r}$$

$$= 0 + 2 + (-4) + 0 = -2$$

**Answer (using Green’s Theorem):** Let $D = [0, 2] \times [0, 1]$. Note that this region is inside the curve $C$. However, there is a subtle problem: the boundary $\partial D$ is not quite the same as $C$; it is the same curve, but with the opposite orientation. So, $\partial D = -C$. Then Green’s theorem tells us that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = - \iint_D \mathbf{F} \cdot d\mathbf{r} = - \iint_D (2x - 2y) \, dA.$$

This last integral is easy to evaluate:

$$- \iint_D (2x - 2y) \, dA = - \int_0^1 \int_0^2 (2x - 2y) \, dx \, dy = -(2) = -2.$$

As predicted, the same value.

**Key Ideas: Curl and Divergence**

- We have seen that for a function $f$, $\nabla f$ means “the gradient of $f$.” However, there seems to be more to this $\nabla$ guy than we have seen so far. In fact, if you look at research papers that apply multivariable calculus, he also seems to be dotted with and crossed with things. (Note: I don’t mean that $\nabla f$ is dotted and crossed, just $\nabla$!) This would lead us to suspect that $\nabla$ itself is a vector.

- **Definition:** Let $\nabla$ denote the vector differential operator

$$\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}.$$
Then it continues to be true that $\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$. Now, however, we can see this as the scalar product of the vector $\nabla$ with the scalar function $f$.

- **Curl and Div:** For a vector, there are three sorts of products: scalar, dot and cross. What do we get when we use these other products with $\nabla$?

**Definition:** For $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ (a vector field on $\mathbb{R}^3$) we define

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} \quad \text{div } \mathbf{F} = \nabla \cdot \mathbf{F}.$$  

Or, writing them out in terms of their components,

$$\text{curl } \mathbf{F} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}$$

and

$$\text{div } \mathbf{F} = P_x + Q_y + R_z.$$ 

**Question 1:** Let $\mathbf{F} = \langle xz, xy^2z^3, x - e^z \rangle$. Find curl $\mathbf{F}$ and div $\mathbf{F}$.

- **How does this relate to what I said yesterday?** If $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} + 0\mathbf{k}$ (that is, if it is really a 2-dimensional vector field with no dependence on $z$ and no vertical component) then curl $\mathbf{F} = (Q_x - P_y)\mathbf{k}$.

- **Physical Significance of curl:** As in the two-dimensional case, curl $\mathbf{F}$ tells us something about how much “spinning” there is in the vector field if interpreted as the velocity field of some fluid. In particular, one way to construct the vector field curl $\mathbf{F}$ from the vector field $\mathbf{F}$ is the following: At each point, drop a little “x”. Orient the $x$ into a position where it spins most rapidly. Make a vector which points in a direction perpendicular to that spinning so that if you look down at the $x$ from the head of the vector you see it spinning counter-clockwise. The length of the vector measures the speed of the spinning. This is an important idea in fluid mechanics where we can literally think of the curl as representing the fluid spinning around, but in fact the idea is very useful in applications to math and physics even when this interpretation does not apply.

- **Physical Significance of div:** If we think of $\mathbf{F}$ as the velocity field of a fluid, then div $\mathbf{F}$ is a scalar which indicates whether the fluid is becoming compressed or expanded at each point. If the vectors all point away from the point $(x, y, z)$ then the divergence there would be positive. If the vectors all point towards a point, the divergence would be negative. If the water is heating up and so expanding, then the divergence would be positive. If it is being sucked up by a black hole, it would be negative. (Often, just to make things simpler – and it really does – people working in fluid dynamics choose to assume that the velocity field is “divergence free” and justify it by saying that the fluid is “incompressible”.)

- **Our main** interest in div and curl is in seeing them applied to theorems analogous to Green’s theorem. However, we will not see that today. Until we do, we can use them to identify conservative vector fields (which is useful for applying FTC) and identifying vector fields that are the curl of some other field (though at the moment you may wonder why you would care).

- **Curl Kills Grad:** If you start with a nice function, take its gradient and then take the curl of that, you will get 0. Moreover, if you start with a vector field and find that its curl is zero, then (given some topological requirements) you can conclude that the vector field is conservative!
**Theorem:** For any function \( f(x, y, z) \) with continuous second partial derivatives, \( \text{curl} \left( \nabla f \right) = \langle 0, 0, 0 \rangle \). Moreover, if \( F \) is a vector field defined on an open, simply connected region in space and \( \text{curl} F = \langle 0, 0, 0 \rangle \) there then there exists a function \( f(x, y, z) \) such that \( \nabla f = F \) at each point in that region.

**Question 2:** Let \( f(x, y, z) = \sin(xy) + e^z \). Find \( F = \nabla f \) and \( \text{curl} F \).
- This provides us with an effective means to determine whether a vector field is conservative in 3 dimensions analogously to our \( P_y = Q_x \) check in 2 dimensions.

**Question 3:** Determine whether the vector field \( F = \langle 2xy^2 - 3x^2z, 2x^2y, -x^3 \rangle \) is conservative.
- **Div Kills Curl:** A similar (and equally surprising) fact is that if you take the curl of a nice vector field and then take the divergence of that, you will get zero. This gives us a way to check if a vector field is the curl of some other vector field. If the divergence is not zero, then it can’t be a curl since:

**Theorem:** If \( F \) is a vector field on \( \mathbb{R}^3 \) whose component functions have continuous second-order partial derivatives then
\[
\text{div curl} \ F = 0.
\]

**Question 4:** Which of these vector fields is the curl of another vector field?
\[
\langle x, -y, z \rangle \quad \langle x, 2y, -3z \rangle
\]
- Note that \( \text{grad} \ f \) and \( \text{curl} \ F \) are vector fields while \( \text{div} \ F \) is a scalar. This means that certain combinations of these things make sense while others do not. For instance, we know now that \( \text{curl} \left( \text{grad} \ f \right) \) not only makes sense, it is the zero vector. But, \( \text{grad} \left( \text{curl} \ f \right) \) makes no sense at all since you can’t take the curl of a scalar function or the gradient of a vector field!
- All of this starts to seem a bit mysterious. Why does curl kill grad and div kill curl? And what kills div? These questions (like the question of when a seemingly conservative vector field will have a non-zero integral around a closed loop) lead us directly to deep mathematical subjects like cohomology. For those interested, I will explain it in class one day after the fourth exam (and promise that it will not be on the final). Those who attend that will see that div, grad and curl are more alike than they may at first appear, and all three can be derived from the same formula.

**Homework**

**Section 16.5:** Read the section carefully and do problems 1–3, 4\(^*\), 5–8, 12\(^*\), 13–20.