Key Ideas: Green’s Theorem

- We have (I’m sure you will recall) integrals over lines and curves and integrals over two-dimensional regions in the plane (such as the interior of a disk or the space between two parabolas). How are these two types of integrals related to each other? It is through a theorem known as Green’s Theorem, which I think you should really try to think of as the FTC for double integrals, as I will explain.

- **Closed curves as boundaries of regions:** Let $C$ be a closed curve in the plane and $D$ be the region that it bounds. We will here always assume a counter-clockwise orientation for $C$ which I will call the natural orientation for a planar region. (Note: the book calls it “positive orientation”...but this terminology seems to confuse students and I’m going to try to avoid it this semester.)

- **The boundary of a given region:** We often will want to think of this relation between $C$ and $D$ beginning in the other direction, although then it could be a little more complicated. Let $D$ be a closed, bounded region in the plane. Then $C = \partial D$ (read “boundary of $D$”) is the curve (or, are the curves...there may be more than one) at the edge of $D$ oriented so that as you walk around the curve, the region always stays on your left.

  Why do I say that the second note about boundaries above is more complicated than the first? Usually, there is no difference. If $D$ is a disk then $\partial D$ is the circle at its edge with the counter-clockwise orientation. However, what if $D$ is the region between two concentric circles (which you will note is not simply connected)? Then $\partial D$ is two circles, and one is oriented counter-clockwise while the other is oriented clockwise.

- **Curl:** Given a vector field in the plane, $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$, we define the “curl of $\mathbf{F}$” to be a kind of derivative of the vector field that has scalar values:

  $$\text{curl } \mathbf{F} = Q_x(x, y) - P_y(x, y).$$

  Loosely speaking, this number actually measures the “rotation” of the vector field around the point $(x, y)$, so the name is appropriate. It is positive if the field rotates counter-clockwise around $(x, y)$, negative if it rotates clockwise, and is zero if there is no rotation.

**Green’s Theorem:** For the vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$

$$\int \int_D \text{curl } \mathbf{F} \, dA = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}.$$  

Or, writing the same thing in terms of $P$ and $Q$:

$$\int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_{\partial D} P \, dx + Q \, dy.$$
• So, Green's theorem says that integrating a curl of a vector field over \( D \) is the same as integrating the vector field around \( \partial D \)!
• Notice that this has a similar structure to both of the FTC's we know. In particular, an integral of some integrand that is “a derivative” turns out to be the “undifferentiated thing” evaluated on the boundary. (Here, unlike the other cases, the boundary is not just isolated points, so “evaluated” comes to mean integration itself, but of a lower-dimensional type.)
• **Proof of Green’s Theorem:** The proof of the theorem (in the case of a particularly simple type of region \( D \)) is shown in the book on page 1056. It is not especially difficult, but also not particularly revealing and so I will not spend much, if any, class time on it. But, I can explain Green's theorem in terms of “curl” and why it makes sense that it is true!

**Question 1:** Use Green's Theorem to evaluate
\[
\oint_C (3y - e^{\sin(x)}) \, dx + (7x + \sqrt{y^4 + 1}) \, dy
\]
where \( C \) is the circle of radius 3 centered at the origin. (Note: the notation \( \oint_C \) is often used for line integrals around closed curves with positive orientation, but it really has no particular meaning other than \( \int_C \) as we have already defined it.)

• **The other way...:** In the example above, we used Green's Theorem to turn a line integral into a simpler-to-compute double integral. Sometimes, however, we will use it in the other direction. This can be done, for example, to compute an area as a line integral.

**Question 2:** Verify that if \( D \) is a region with boundary \( C = \partial D \) then the area \( A \) of \( D \) can be computed by any of the following line integrals
\[
A = \oint_C x \, dy = -\oint_C y \, dx = \frac{1}{2} \oint_C x \, dy - y \, dx.
\]

**Question 3:** Use the third formula above to compute the area of the ellipse parametrized as \( r(t) = (a \cos t, b \sin t) \) for \( 0 \leq t \leq 2\pi \). (This is the ellipse \( x^2/a^2 + y^2/b^2 = 1 \).)

• Note that this idea for computing areas by integrating around the boundary is not just “abstract mathematics”. There is a machine called a planimeter which measures areas using this method!
• Look at what happens to Green's Theorem in the case of a conservative vector field! If \( \mathbf{F} \) is conservative then the curl is zero \( (Q_x - P_y) = 0 \) and so the double integral is obviously zero. In fact, although I did not mention this before, this is part of the proof of the fact that \( P_y = Q_x \) implies that a field is conservative.

• **PS:** By the way, you might want to look at the brief biography of George Green on page 1056. He used this theorem in a paper on electro-magnetism back in 1828, without having ever attended college! The significance of this result was not noticed until 1846, and his ideas of electro-magnetism were later developed by others (including Maxwell) into our present theory of light and radio waves.

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**Homework**

**Section 16.4:** Read the section carefully and do problems 1, 2*, 3–9, 11, 12*, 13, 14