Key Ideas: FTC for Line Integrals

- Recall what it means to say that a vector field is conservative. We say $F$ is conservative if there is a function (the potential function) $f$ such that $F = \nabla f$.

- Actually, it is a bit more subtle than it seems at first. In this section, it is necessary for us to talk about where it is conservative. A vector field can be conservative in one place but not conservative in another. To avoid getting bogged down in this subtle detail, I will first consider the situation in which a vector field is conservative everywhere and then we will reconsider the situation in which it is only conservative in some places later.

- The Fundamental Theorem of Calculus for Line Integrals: If $C$ is an oriented path that starts at the point $a$ and ends at the point $b$ and $F = \nabla f$ is a conservative vector field then

$$\int_C F \cdot dr = f(b) - f(a).$$

- Note that the textbook writes this slightly differently, emphasizing that we have parametrized $C$ by $r$ so that they write $r(b)$ where I write $b$ and similarly for $a$. However, I prefer my notation because it emphasizes the importance of the points themselves in this theorem. Still the other version is apparent in the proof of this theorem:

- Proof of FTC for Vector Fields: Let $r(t)$ parametrize $C$ (in the right direction) for $a \leq t \leq b$. (So, for instance $b = r(b)$.) Then note that $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t)$. Consequently,

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) dt = \int_a^b \frac{d}{dt}f(r(t)) dt.$$

So, the usual FTC for functions of one variable ($t$ in this case) gives that this is the same as $f(r(b)) - f(r(a))$.

- The 1-dimensional version as a special case: In fact, you can view the FTC that you learned in Calc 1 as being a special case of this theorem. It is the special case that the vector field is 1-dimensional and the curve is an interval in the $x$-axis. However, there is a very important difference that must be observed. The 1-dimensional version of the FTC works for any function $f(x)$ because every function has an antiderivative. In contrast, this only works for conservative vector fields, and most vector fields are not conservative!

**Question 1:** Let $f(x, y, z) = xy^2e^z$. Find its gradient vector field $F = \nabla f(x, y, z)$. Compute the integral of this vector field along the path mapped out by $r(t) = \langle \cos(\pi t), t + 1, 5t - 5t^2 \rangle$ for $0 \leq t \leq 1$. 
Independence of Path: Notice that in the previous question, the particular path that was chosen had nothing to do with the answer. In particular, since the endpoints were the only significant thing about the path, we could have used any path connecting those points and found the same answer. This is a very important observation:

**Theorem:** If the vector field \( \mathbf{F} \) is conservative (in a simply connected region \( D \)), then the value of a path integral of \( \mathbf{F} \) (along a path in \( D \)) depends only on the endpoints of the curve and not on the curve itself.

I will explain later about this “simply connected” condition, which is a necessary but picky little detail. In general, we will just summarize the main idea simply by saying: Line integrals of conservative vector fields are independent of path.

Note that this is not true for non-conservative vector fields. It is easy to come up with an example for which the value of the path integral between two points depends on the particular choice of path.

**Question 2:** Compare the integral of \( \mathbf{F} = \langle -y, x \rangle \) along three paths that start at \((1, 0)\) and go to \((-1, 0)\): counter-clockwise around unit circle, clockwise around unit circle and straight across!

**Integrals around a closed loop:** Another easy consequence of the FTC for line integrals is that if a vector field is conservative (in a simply connected region \( D \)) and \( C \) is a closed loop (in \( D \)), then the integral of that vector field around the closed loop \( C \) will be zero. (This is because in that case \( b \) and \( a \) would be the same!)

Because of this theorem, it is clearly important to be able to tell when a vector field \( \mathbf{F} \) is conservative and to be able to find a potential function \( f \) if it is. First, we note that path independence has a special meaning in terms of paths that start and end at the same point.

**Theorem:** The line integral of a vector field \( \mathbf{F} \) is independent of path (in a region \( D \)) if and only if its integral around any closed loop (in \( D \)) is zero.

**Proof:** The proof goes in two directions. First notice that if it is independent of path, then the integral around a closed loop would have to be zero because it would be the same as the integral over the path of length zero that just stays at the same point. Thus, independence of path implies integrals around closed loops must be zero. However, we have to check that the implication works in the other direction as well. Suppose we know that the integral around any closed loop is zero. Then if \( C_1 \) and \( C_2 \) are two paths with the same start and endpoint, I can show the integrals along these must be equal because \( C_1 \) followed by \(-C_2\) is a closed path and hence

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0.
\]

**Question 3:** Use the preceding theorem to prove that \( \mathbf{F} = \langle y, -x \rangle \) is not conservative. (Hint: Try integrating it around the unit circle \( x^2 + y^2 = 1 \) starting at \((1, 0)\) and going counter-clockwise.)

**Some topology:** In order to be able to answer the most interesting question (“How can we tell if a given vector field is conservative?”) we must learn a few simple terms from topology. A region \( D \) in the plane (or in space) is said to be:

- **open** if every point in it can be surrounded by a disk (or sphere) that stays entirely in \( D \).
- **connected** if every two points in it can be connected by a path that stays entirely in \( D \).
simply connected if $D$ is connected and any closed loop in $D$ does not enclose any points outside of $D$. (This last one can be thought of as saying that $D$ has no “holes” in it.)

- **Theorem:** If $\mathbf{F}$ is a continuous vector field on an open connected $D$, then it must be conservative if it is independent of path. (In other words, in such a case, you can check to see if it is independent of path, or equivalently if the integral around any closed loop is zero, and the answer is yes if and only if $\mathbf{F}$ is conservative.)

- It is easy for us to conclude from earlier results on partial derivatives that if $\mathbf{F}$ is a conservative vector field and the derivatives of its component functions are continuous, then the $y$-derivative of the first component must equal the $x$-derivative of the second component. (This is just the equivalence of second order mixed partial derivatives.) This is the test we will use to determine whether vector fields in the plane are conservative (even though it is not the strongest method that this section covers).

**Question 4:** Check whether $\mathbf{F}(x, y) = (x - y)i + (x - 2)j$ is conservative.

- More importantly, we can also state the converse of this rule if the region is nice enough:

  **Theorem:** If $\mathbf{F} = Pi + Qj$ is a vector field in an open simply-connected region $D$ and at each point in $D$ we have
  \[
  \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},
  \]
  then $\mathbf{F}$ is conservative in the region $D$.

**Question 5:** Determine whether $\mathbf{F}(x, y) = (3 + 2xy)i + (x^2 - 3y^2)j$ is conservative.

- **Reconstructing the Potential:** Suppose you know (from the above or from some other information) that $\mathbf{F}$ is conservative, how do you find the potential function $f$? There are two different ways to proceed:

  - **Using Line Integrals:** Let $\mathbf{F}$ be a conservative vector field and let $x_0$ and $x$ be two points in its domain. Define the function $f$ by saying that $f(x)$ is the value of the line integral of $\mathbf{F}$ from $x_0$ to $x$. (Along which path? It doesn’t matter since $\mathbf{F}$ is conservative! Choose any path at all.) It turns out that this is a potential function for $\mathbf{F}!$

  - **Using Anti-differentiation:** Alternatively, we can proceed by partial integration finding (up to an unknown function that is the “constant of integration”) the antiderivative with respect to $x$ of $P$ and the antiderivative of $Q$ with respect to $y$. If the vector field is conservative, then these two antiderivatives must actually be equal, and so we just have to determine what those unknown functions might be!

  The first of these is generally used only in theory, while the second one is more useful in practice.

**Question 6:** Find the potential function for the vector field from the previous example. Use it to evaluate the line integral of $\mathbf{F}$ along the arc of the curve $y = x^3$ starting at $(-1, -1)$ and ending at $(1, 1)$.

- Notice that the procedures above for determining whether a vector field is conservative only apply in the case of a vector field on the plane. Conclusive determination of whether a vector field is conservative in space will be addressed in a future lecture. However, the FTC and the procedure for finding the potential function are both true for vector fields in space as well as vector fields in the plane.
A Paradoxical Example: In this class, we will largely be ignoring the scary and difficult examples. As I’ve told you, we’re showing you the petting zoo and rarely let you see the wild beasts. But, you may find this example illuminating. Let

\[ \mathbf{F}(x, y) = \left( \frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2} \right). \]

Looking at a sketch of the vector field, you might think that it is not conservative because it “rotates around the origin” and so we could integrate along \( \mathbf{r}(t) = \langle \cos(t) \sin(t) \rangle \) for \( 0 \leq t \leq 2\pi \) and get something non-zero. On the other hand, we can also check that \( P_y = Q_x \) which suggests that it is conservative! What’s going on?

Explanation: This is where the topology stuff becomes important. You must note that \( \mathbf{F} \) is undefined at \((0, 0)\). Hence, it is true that \( \mathbf{F} \) is conservative in any simply connected region (which means inside any circle that does not contain the origin) and in any such region the integral around a closed loop is zero. Also, in any such region we can make a potential function using the more theoretical procedure, even though you will not be able to find a formula for a potential function which is defined on the entire domain of \( \mathbf{F} \). This is deep and important stuff that becomes immensely useful in analysis (where it develops into the notion of Riemann Surfaces and a generalization of the notion of “function”) and in topology (where we are able to identify holes in spaces by recognizing such situations).

Homework

Section 16.3: Please read this section and do problems 1, 3–11, 12\(^*\), 13, 14\(^*\), 15, 16\(^*\), 17–20, 23