Key Ideas: Path Integrals (continued)

- Do the students have any questions from the homework assigned last time?
- Recall from last time the definitions of the integrals
  \[ \int_{C} f \, ds \quad \text{and} \quad \int_{C} \mathbf{F} \cdot d\mathbf{r}. \]

The first is the integral of a function over the curve \( C \) (integrated with respect to arclength). The latter is the integral of a vector field over the curve \( C \).

- **Important Observation: Orientation Matters:** One big difference between integrals of functions with respect to arclength and integrals of vector fields is that the second depends not only on the curve but also on the direction that you sweep it out in your parametrization. So, a question involving a path integral of a vector field must specify this orientation of the curve as well as the set of points that make it up.

- **Reversal of Orientation:** If we define \(-C\) to be the same curve as \(C\) but swept out in the other direction, we find that the integrals over \(C\) and \(-C\) have opposite signs while the integral with respect to arclength over these two curves are equal. In symbols, we can write
  \[ \int_{-C} \mathbf{F} \cdot d\mathbf{r} = - \int_{C} \mathbf{F} \cdot d\mathbf{r}, \quad \int_{-C} f \, ds = - \int_{C} f \, ds. \]

The next example gives a physical interpretation to this fact.

**Question 1:** Find the work done by the force field \( \mathbf{F}(x, y) = \langle x^2, -xy \rangle \) in moving a particle along the arc of the circle \( x^2 + y^2 = 1 \) from \((1, 0)\) to \((0, 1)\). What if you went along the path in the opposite direction? Why does this make sense?

- **Breaking Up:** If the curve \(C\) is actually described as the union of a bunch of different curves \(C_1, C_2, \text{ etc.} \) each of which is smooth, then we say \(C\) is **piecewise-smooth** and we define the integral over \(C\) just to be the sum of the integrals over the individual pieces.

**Question 2:** Evaluate the line integral of \( f(x, y) = 2x \) along the curve \(C\) which is the arc of the parabola \( y = x^2 \) from \((0, 0)\) to \((1, 0)\) followed by the vertical line segment from \((1, 1)\) to \((1, 2)\).

(The above example illustrates the idea of breaking up the curve into smaller pieces in the case of an integral with respect to arclength, but the same idea applies to integrals of vector fields as well. If there is time, we can integrate \( \mathbf{F} = \langle x + y, 1 \rangle \) over the same path \(C\) as in the previous question.)
Another Notation: If \( \mathbf{F}(x, y) = (P(x, y), Q(x, y)) \) then we can also write the integral of this over \( C \) as

\[
\mathbf{F} \cdot d\mathbf{r} = P
dx + Q
dy.
\]

(Or if it is a 3-dimensional vector field, \( \int_C P
dx + Q
dy + R
dz \).) In fact, the book provides a separate definition for these things, but it turns out just to be the same in the end and so we can just introduce it as an alternative notation. Note that if the \( dx \) term is missing, that just means \( P \equiv 0 \) and similarly if one of the other differentials is missing.

**Question 3:** What vector field is being integrated in \( \int_C (x^2 - z)
dy \) where \( C \) is the straight line connecting \((0, 1, 2)\) and \((-8, 3, 7)\)?

- In a later section, it will be important to be able to integrate a vector field around the boundary of a surface. This requires you to figure out what the boundary is and to parametrize it:

**Question 4:** Let the curve \( C \) be the boundary of the part of the graph of the function \( f(x, y) = x^2 + y^2 \) which lies beneath the plane \( z = 5 \), oriented so that it is traversed in a counter-clockwise direction when viewed from above. Compute the path integral of the vector field \( \mathbf{F}(x, y, z) = (yz, -xz, e^{x^2}y) \) along the path \( C \).

- Note homework assignment below. Problems with “stars” will be collected and graded on Tuesday.

**Homework**

**Section 16.2:** Please read this section and do problems 7, 8 , 15, 16