Key Ideas: FTC for Line Integrals I

- **The Fundamental Theorem of Calculus for Line Integrals:** If $C$ is an oriented path that starts at the point $a$ and ends at the point $b$ and $F = \nabla f$ is a conservative vector field then

  \[ \int_C F \cdot dr = f(b) - f(a). \]

- Note that the textbook writes this slightly differently, emphasizing that we have parametrized $C$ by $r$ so that they write $r(b)$ where I write $b$ and similarly for $a$. However, I prefer my notation because it emphasizes the importance of the points themselves in this theorem. Still the other version is apparent in the proof of this theorem:

- **Proof of FTC for Vector Fields:** Let $r(t)$ parametrize $C$ (in the right direction) for $a \leq t \leq b$. (So, for instance $b = r(b)$.) Then note that $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t)$. Consequently,

  \[ \int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) \, dt = \int_a^b \frac{d}{dt} f(r(t)) \, dt. \]

  So, the usual FTC for functions of one variable ($t$ in this case) gives that this is the same as $f(r(b)) - f(r(a))$.

- **The 1-dimensional version as a special case:** In fact, you can view the FTC that you learned in Calc 1 as being a special case of this theorem. It is the special case that the vector field is 1-dimensional and the curve is an interval in the $x$-axis. However, there is a very important difference that must be observed. **The 1-dimensional version of the FTC works for any function $f(x)$ because every function has an antiderivative. In contrast, this only works for conservative vector fields, and most vector fields are not conservative!**

**Question 1:** Let $f(x, y, z) = x y^2 e^z$. Find its gradient vector field $F = \nabla f(x, y, z)$. Compute the integral of this vector field along the path mapped out by $r(t) = (\cos(\pi t), t + 1, 5t - 5t^2)$ for $0 \leq t \leq 1$. 
• **Independence of Path:** Notice that in the previous question, the particular path that was chosen had nothing to do with the answer. In particular, since the endpoints were the only significant thing about the path, we could have used any path connecting those points and found the same answer. This is a very important observation:

**Theorem:** If the vector field $\mathbf{F}$ is conservative, then the value of a line integral of $\mathbf{F}$ depends only on the endpoints of the curve and not on the curve itself. *Line integrals of conservative vector fields are independent of path.*

Note that this is not true for non-conservative vector fields. It is easy to come up with an example for which the value of the path integral between two points depends on the particular choice of path.

**Question 2:** Compare the integral of $\mathbf{F} = \langle -y, x \rangle$ along three paths that start at $(1, 0)$ and go to $(-1, 0)$: counter-clockwise around unit circle, clockwise around unit circle and straight across!

• **Integrals around a closed loop:** Another easy consequence of the FTC for line integrals is that if you integrate a conservative vector field around a closed loop, the value of the integral will be zero. (This is because in that case $b$ and $a$ would be the same!) It is a useful example to prove that this is actually just another way to say that the integrals are independent of path.

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**Homework**

**Section 16.3:** Please read this section and do problems 1, 11, 12*, 13, 17

**Hints:** For #11 note that if $f(x, y) = x^2y$ then $\nabla f = \langle 2xy, x^2 \rangle$. For #12 note that if $f(x, y) = (x^3 + y^3)/3$ then $\nabla f = \langle x^2, y^2 \rangle$. For #13, use $f(x, y) = (x^2y^2)/2$ because then $\nabla f = xy^2 \mathbf{i} + x^2y \mathbf{j}$. And for #17 note that $f(x, y, z) = xy^2 \cos(z)$ is a potential function for $\nabla f(x, y, z) = \langle y^2 \cos z + 2xy \cos z, xy^2 \sin z, 0 \rangle$. 