Key Ideas: Line (Path) Integrals

- In earlier calculus classes, you learned how to integrate a function of one variable along a piece of a line segment. This integral had the interpretation of being the difference between the area above the axis and below the graph and the area below the axis and above the graph. A generalization of this same idea is integrating a function along a curve in the plane or a curve in space. Ironically, such an integral is known as a “line integral” even though the most interesting thing about it is that the curve you are integrating over need not be straight. In fact we will be learning about three different sorts of line integrals, their interpretations and relationships.

- Integrating a function along a curve: Let $C$ be a curve in the plane and $f(x, y)$ be a function whose domain includes $C$. Then we define

$$
\int_C f(x, y) \, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i) \Delta s_i
$$

to be the limit of sums of values of the function at points that divide $C$ into $n$ pieces multiplied by the length of those pieces as $n$ goes to infinity. We call this the line integral of $f$ along $C$ with respect to arclength. (Remember the parameter $s$ which measures the arclength that we introduced earlier in the course.) If $C$ happens to be a piece of the $x$ axis then this just reproduces the usual integral.

- How to do it practically: As before, the use of $s$ is simple conceptually but difficult to actually use in practice. Consequently, we pick any smooth parametrization of $C$ (that is, $C$ is given by $r(t) = (x(t), y(t))$ for $a \leq t \leq b$ so that $r'(t) \neq 0$). Then since we know that

$$
\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}
$$

one can determine that

$$
\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.
$$

(Note that this translates the line integral into an “ordinary” integral along the $t$ axis. You can even view it as a “change of coordinates” with the introduction of an appropriate Jacobian!)
• **Geometric Interpretation:** The good news is that there is a simple geometric interpretation of what this means. (The bad news, as usual, is that we will later generalize the notion of a line integral to other situations in which the geometric interpretation disappears!) If the function \( f(x, y) \) is positive all along \( C \) then the arclength integral along \( C \) gives the area of the curvy fence (or curtain?) that has height \( f(x, y) \) above each point \((x, y)\) on the curve. More generally, it is the area of the part of the fence over the \( xy \)-plane minus the area of the part below.

• **Breaking Up:** If the curve \( C \) is actually described as the union of a bunch of different curves \( C_1, C_2, \) etc. each of which is smooth, then we say \( C \) is **piecewise-smooth** and we define the integral over \( C \) just to be the sum of the integrals over the individual pieces.

**Question 1:** Evaluate the line integral of \( 2x \) along the curve \( C \) which is the arc of the parabola \( y = x^2 \) from \((0, 0)\) to \((1, 0)\) followed by the vertical line segment from \((1, 1)\) to \((1, 2)\).

• **Line Integrals in Space:** Everything I've said above applies just as well to integrals along curves in space. In particular we can say

\[
\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} \, dt.
\]

• **Integrating Vector Fields:** Let \( F \) be a vector field defined on a smooth curve \( C \) which is parametrized as \( r(t) \) for \( a \leq t \leq b \). Then we define

\[
\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) \, dt.
\]

• **Relating the Different Types of Integrals:** If \( T = r' / |r'| \) is the unit tangent vector to \( C \) then

\[
\int_C F \cdot dr = \int_C F \cdot T \, ds.
\]

We do not use this formula often, but it is important in understanding what the integral of a vector field **tells us.** What it says is that integrating a vector field along a curve is the same as “adding up” the numbers \( F \cdot T \) at each point, which measure how much the vector field \( F \) points in the same direction as the parametrization of the curve. (Recall that \( v \cdot w \) is largest when the two vectors point in the same direction, zero when they are perpendicular, and negative when the angle they form is obtuse!)

• **A Physical Interpretation:** If \( F \) is a force field, then the line integral computes the amount of work done by the field in moving an object along the curve. A good example to imagine is pushing a heavy sailboat along a sandy path on a windy day. If the wind is in the right direction, it will be doing some of the work for you. So, the integral of the force field that the boat will feel from the wind along the path is the amount of work that the wind did. (You probably did some work too, but less than you would have if you had not had the wind's help.) On the other hand, if you are trying to push it along the path in the opposite direction, the wind will make it harder for you. So, the value of the integral is negative, indicating that the
contribution of the wind was to make you have to do that much more work than you would have had to have done on a windless day.

**Question 2:** Evaluate \( \int_C \langle y^2, x \rangle \cdot dr \) where \( C \) is the straight line segment from \((-5, -3)\) to \((0, 2)\). Compute the same integral where \( C \) is the arc of the parabola \( x = 4 - y^2 \) from \((-5, -3)\) to \((0, 2)\). (See example 4 on page 1038.)

- **Important Observation: Path Dependence:** Usually, the integral of a vector field depends not only on the endpoints but on the particular curve being integrated over. We will learn later about a special situation in which the endpoints are all that matter, but in general, the curve makes a difference.

- **Important Observation: Orientation Matters:** One big difference between integrals of vector fields and integrals of functions with respect to arclength is that the first depends not only on the curve but also on the direction that you sweep it out in your parametrization. So, if we define \(-C\) to be the same curve as \(C\) but swept out in the other direction, we find that the integrals over \(C\) and \(-C\) have opposite signs (while the integral with respect to arclength over these two curves are equal). The next example gives a physical interpretation to the fact that \( \int_C F \cdot dr = - \int_{-C} F \cdot dr \).

**Question 3:** Find the work done by the force field \( F(x, y) = \langle x^2, -xy \rangle \) in moving a particle along the quarter circle from \((1, 0)\) to \((0, 1)\). What if you went along the path in the opposite direction?

- **Another Notation:** If \( F(x, y) = \langle P(x, y), Q(x, y) \rangle \) then we can also write the integral of this over \(C\) as

\[
\int_C F \cdot dr = \int_C P \, dx + Q \, dy.
\]

(Or if it is a 3-dimensional vector field, \( \int_C P \, dx + Q \, dy + R \, dz \).) In fact, the book provides a separate definition for these things, but it turns out just to be the same in the end and so I'll just introduce it as an alternative notation. **Note that if the \( dx \) term is missing, that just means \( P \equiv 0 \) and similarly if one of the other differentials is missing.**

**Question 4:** What vector field is being integrated in \( \int_C (x^2 - z) \, dy \) where \( C \) is the straight line connecting \((0, 1, 2)\) and \((-8, 3, 7)\)?

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**Homework**

**Section 16.2:** Please read this section and do problems 1–4, 7–8, 15–16, 17–18, 19–22