Key Ideas: Change of Coordinates – a useful and unifying idea

- **Change of Variables as “Transformations”:** We have now studied several instances of “changes of variables” and their applications to computing integrals. For instance:
  - In calc 1 you saw what happened if you said $u = f(x)$, $du = f'(x)dx$.
  - Earlier this year we let $x = r \cos \theta$ and $y = r \sin \theta$ allowing $dA = r \, dr \, d\theta$.
  - Finally, we have now seen spherical coordinates for which $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$, along with its more complicated looking formula for $dV$.

I would like us to be able to fit all three of these together into a single observation that would allow us to do even weirder changes of variables if we ever would need to.

- So, what do these all have in common? In each case, we take our $n$-dimensional space (the line, the plane or 3-space) and replace the variable with $n$ functions of $n$ variables each. This suggests to me the following definition:

  **Transformation:** We consider $u = f(x)$ to be a transformation from the line with variable $x$ to the line with variable $u$. We consider $x = g(u, v)$ and $y = h(u, v)$ to be a transformation from the plane with coordinates $u$ and $v$ to the plane with coordinates $x$ and $y$. Finally, we consider $(x, y, z) = (g(u, v, w), h(u, v, w), k(u, v, w))$ to be a transformation from the space with coordinates $(u, v, w)$ to the space with coordinates $(x, y, z)$.

- We will generally assume that these transformations are “smooth”, which means that $f$, $g$, $h$ and $k$ all have continuous first derivatives in each of the variables.

- **One-to-Oneness:** We say that a transformation is one-to-one when taking a region $S$ onto a region $R$ if no two points in $S$ get sent to the same point in $R$.

  **Question 1:** If $S = \{(\rho, \theta, \phi) | 1 \leq \rho \leq 2, a \leq \theta \leq b, -\pi/2 \leq \phi \leq \pi/2\}$, is the transformation associated to spherical coordinates one-to-one when $a = 0$ and $b = 2\pi$? Are there other choices of $a$ and $b$ for which it is one-to-one? Name some examples.

- **The Jacobian:** An important object here is the Jacobian, which is the determinant of the matrix of all of the first derivatives of the functions defining the transformation. So, in the one dimensional case, it is just $f'(x)$. However, for 2 and 3-dimensions we have

\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad \frac{\partial (x, y, z)}{\partial (u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.
\]
**Question 2:** Find the Jacobian of the transformation \( x = u + v, \ y = u - v. \)

**Question 3:** Find the Jacobian for the transformations associated to polar and spherical coordinates.

**Main Result:** The absolute value of the Jacobian is the thing that shows up when transforming integrals:

\[
\iint_R f(x, y) \, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv
\]

\[
\iiint_R f(x, y, z) \, dV = \iiint_S f(x(u, v, w), \ldots, z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du \, dv \, dw.
\]

An important thing to keep in mind in these transformations is seeing what becomes of some region \( S \) in the original space when we transform it (in a one-to-one fashion) to a region \( R \) in the new space, or vice versa. (Hint: You can usually do this algebraically by seeing what happens to the equations that define the boundary of the region under appropriate substitutions.) We are especially interested in the case where a messy region \( R \) corresponds to a nice “rectangular” region \( S \) in the \( uv \)-plane. (Note: We’ve been doing this with polar coordinates for a week or so now!)

**Question 4:** A transformation is defined by the equations \( x = u^2 - v^2, \ y = 2uv. \) Find the image of the square \( S = [0, 1] \times [0, 1] \) in the \( uv \)-plane. What does it look like in the \( xy \)-plane?

**Question 5:** Consider the parallelogram \( P \) bounded by the lines \( y = x/2, \ y = 2x, \ y = (x + 3)/2 \) and \( y = 2x - 3. \) Verify that this is the image of a rectangle under the transformation \( x = 2u + v \) and \( y = u + 2v. \) Use this fact to integrate function \( f(x, y) = 3x + 2y \) over \( P. \)

**Question 6:** Consider the transformation given by \( x = u/v \) and \( y = v. \) What region \( S \) in the \( uv \)-plane gets sent to the region \( R \) in the first quadrant bounded by the lines \( y = x, \ y = 3x, \ xy = 1 \) and \( xy = 3? \) Use this information to evaluate the integral of the function \( f(x, y) = y \) over \( R. \)

---

**Homework**

**Section 15.9:** Please read this section and do problems 1, 2, 3–6, 7–8, 11, 12, 13–16

**Test Next Week:** No “starred” problems appear above because our third test will be given next Tuesday, November 3rd.