Key Ideas: Vector Fields

- **Vector Fields:** A function \( \mathbf{v} = \mathbf{f}(x, y) \) that takes points \((x, y)\) in the plane to vectors in the plane is called a vector field. Similarly, a function \( \mathbf{v} = \mathbf{f}(x, y, z) \) that takes points \((x, y, z)\) in space to 3-component vectors is called a vector field.

- The way that we picture a vector field is to imagine seeing a representative of the vector \( \mathbf{f} \) situated at the point \( x \). Note, however, that we may scale all of the lengths up or down (simultaneously) to make the picture easier to understand. (That is, it is their relative lengths that are really important. We try to make sure the vectors are not so long that they overlap or so short that they are hard to see.)

**Question 1:** Sketch the vector field \( \mathbf{f}(x, y) = -yi + xj \) by drawing the vectors at all points with integer coordinates in \([-1, 1] \times [-1, 1]\).

- As in the example above, you can always view a vector field as being made up of scalar fields which are the individual components of the field.

- **Examples of Vector Fields:** There are many examples of vector fields that show up in math and science. Here are just a few examples:
  - The gradient \( \nabla f \) is actually a vector field associated to any function \( f \). Remember, it associates to each point the vector that points in the direction of greatest increase of \( f \) with length equal to the directional derivative in that direction. It has the property that the vectors in the field are always orthogonal to the level curves (or surfaces).
  - Imagine a wind storm. At each point in the storm, the wind has a direction and a velocity. If we associate these velocity vectors to each point, we have a vector field that gives you an idea of what direction the wind would push and with what force at each point.
  - Similarly, a gravitational field produces a force acting on a mass placed at any point in the field. We can associate to the point the corresponding force vector, giving a vector field that indicates the effect of gravity at each point. (The gravitational field of the Earth would look like a bunch of vectors pointing towards the center of the planet which get shorter and shorter the farther away you go.

- Given a vector field \( \mathbf{F} \), we can ask whether it is the gradient vector of some function \( f \). In fact, it turns out that this is a really important question in many areas of math and theoretical science. Inspired by physical motivations that we will hopefully discuss later, we say that \( \mathbf{F} \) is conservative if it is the gradient field of some function \( f \). (If it is, we say that \( f \) is the potential function of \( \mathbf{F} \).)

**Question 2:** Find a potential function for the conservative vector field \( \mathbf{F}(x, y) = (3+2xy, x^2-3y^2) \). Why can you not do the same for \( (3 - 2xy, x^2 + 3y^2) \)?

- In a later section, we will learn tests for determining whether \( \mathbf{F}(x, y) \) is conservative and important theorems about integrals that only work for conservative vector fields.

**Homework**

**Section 16.1:** Please read this section and do problems: 1–14, 29–32