Math 221 Handout: October 21, 2011

Alex Kasman
Department of Mathematics
College of Charleston

Question 1: Integrate the function \( f(x, y, z) = x + 2y - z \) over the part of space bounded by the planes \( z = y, z = 2y, x = 0, x = 1 \) and \( z = 1 \).

**Key Ideas: Cylindrical and Spherical Coordinates**

- Sometimes it is useful, when doing a triple integral, to view the coordinates as being polar on one of the coordinate planes and keep the third variable Cartesian. This is known as **cylindrical coordinates**. In particular, we can consider \((r, \theta, z)\) as being cylindrical coordinates for the point \((r \cos \theta, r \sin \theta, z)\). Or, equivalently, we could view the coordinates as being polar in the \(xz\)-plane leaving \(y\) Cartesian. When one does this, you must remember to include an extra factor of \(r\) in the integrand when converting to an iterated integral. (For example, if \(z\) is to be kept Cartesian, you must remember that \(dA = r\, dr\, d\theta\, dz\).)

**Question 1:** Write two iterated integrals for computing the integral of \( \sqrt{x^2 + z^2} \) over the part of space bounded by \( y = 4 \) and \( y = x^2 + z^2 \). (One integral should be in Cartesian coordinates and one should be in cylindrical coordinates.) Evaluate one of the two forms.

- The previous example demonstrates the two “hints” that you want to use cylindrical coordinates: If the region projects onto a polar rectangle in one of the coordinate planes and/or if the function depends on the sum of the squares of two of the variables.

- There is yet another coordinate system that proves useful in computing triple integrals. **Spherical coordinates** \((\rho, \theta, \phi)\) correspond to the usual Cartesian coordinates by the formulas

\[
x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi.
\]

- Note that \(x^2 + y^2 + z^2 = \rho^2\).

- As in the case of polar coordinates, lots of choices of spherical coordinates correspond to the same points in space (which is not nice), but it turns out that these coordinates are nice for describing spheres or cones. In particular, one can think of \((\rho, \theta, \phi)\) as being the point \(\rho\) units out from the origin in the direction of the vector that is \(\phi\) radians down from the \(z\)-axis and \(\theta\) radians counter-clockwise from the \(x\)-axis.

**Question 2:** What part of space do you get when you consider \(a \leq \rho \leq b\) (for positive \(a\) and \(b\))?  

**Question 3:** What part of space do you get when you consider \(a \leq \rho \leq b\) and \(\alpha \leq \theta \leq \beta\)?

**Question 4:** What part of space do you get when you consider \(a \leq \rho \leq b\), \(\alpha \leq \theta \leq \beta\) and \(c \leq \phi \leq d\)?
• This last sort of shape, a *spherical wedge* is the simplest thing to integrate over in spherical coordinates. However, we have to know what happens to $dV$ in terms of these new variables:

$$dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi.$$  

**Question 5:** Evaluate the integral of $\sqrt{x^2 + y^2 + z^2}$ over the unit ball.

• Of course, there are also instances in which more complicated regions in space can be nicely described in spherical coordinates. Consider example 4 on page 1009.

**Question 6:** Use spherical coordinates to find the volume of the “ice cream cone” above $z = \sqrt{x^2 + y^2}$ and below $x^2 + y^2 + z^2 = z$.

• **Important Comment:** When doing a transformation of an integral to polar, spherical or cylindrical coordinates, there are restrictions on what values the variables can take. In particular, $r$ and $\rho$ have to be positive and $\phi$ must be between 0 and $\pi$. (As we will see later, this is because the *absolute value* of the Jacobian is added after the transformation.)

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**Homework**

**Section 15.7:** Read the section and do problems 13, 17–22, 27, 28*

**Section 15.8:** Read the section and do problems 1, 2, 15–16, 19–21, 22*, 23–26

**Note:** Remember I warned you at the beginning of the course that the material gets harder and harder in this course? We’re about to take a “quantum leap” into an area that is much more difficult than what we’ve covered so far. First we will unify the idea of “change of coordinates” that we’ve seen several examples of so far. Then, pretty much all of the previous material will be brought together, along with an entirely new idea called “vector fields”, in a few beautiful and deep theorems. Do not let yourself fall behind! Please be prepared to ask me questions on Monday about anything you may be uncertain about so far.