Key Ideas: Applications of Double Integrals

- Double integrals are useful for more than figuring out volumes associated to graphs. (If that was their only use, then they would have little application in science and engineering. However, as I've shown you in class, they do get used.) To understand this more general use, one should think again of the definition of double integrals in terms of Riemann sums; they involve evaluating the function at a point, multiplying by a little bit of area, and then adding up all of these products. It is almost like adding up all of the values of the function on the region, but that step of multiplying by the little area is key.

- For example, you can read in section 15.5 about how the integral of the density function gives the total mass of an object and the integral of the probability distribution function gives the total probability. A practical example of this sort (for biologists) is the following: If \( f(x, y) \) gives the density of bacteria per unit of area, then the integral of \( f \) over \( R \) gives the total number of cells in the region \( R \). Suppose, for instance, the function \( f \) was constant (something like 125 bacterial cells per square inch). Then multiplying by the area would give the total number of cells, right? Similarly, if \( f \) is not constant, you could get a good estimate of the total number of cells by using that same procedure on little squares (chosen so small that the value of \( f \) does not vary much) and adding up all of these products. In fact, to estimate the number of bacterial cells in a petri dish, my wife who is a molecular biologist essentially does a Riemann sum where the region is broken up into little squares by placing the dish over a piece of graph paper and the "evaluation" of the function is achieved by literally counting cells with a microscope.

- The only application of this sort that we will really consider here is the surface area of part of the graph of \( z = f(x, y) \) in Section 16.6.

Key Idea: Triple Integrals

- Given a function \( f(x, y, z) \) and a box \( B = [a, b] \times [c, d] \times [r, s] \) in space, we can define the triple integral of \( f \) over \( B \) analogously to a double integral. In particular, we break \( B \) down into a collection of smaller boxes of volume \( \Delta V = \Delta x \Delta y \Delta z \) and add up the product \( f(x^*_i, y^*_j, z^*_k) \Delta V \) for some collection of points \((x^*_i, y^*_j, z^*_k)\) (one in each smaller box). Then, taking the limit as these boxes get smaller we get

\[
\iiint_B f(x, y, z) \, dV.
\]

- \textbf{Fubini's Theorem:} As in the lower dimensional cases, we will rarely use the Riemann sums that show up in the definition of the triple integral. (In fact, they are used more often in the "real world" where you may have access to a fast computer than they are in our class.) More often, we will make use of the fact that for a continuous function on \( B \) we have
\[ \iiint_B f(x, y, z) \, dV = \int_{a}^{b} \int_{c}^{d} \int_{r}^{s} f(x, y, z) \, dx \, dy \, dz. \]

- **Note:** The order of integration in the formula above is merely one of six possible orderings. Each will give the same value if the function is integrable on \( B \).

**Question 1:** Evaluate
\[ \iiint_B xyz^2 \, dV \quad B = [0, 1] \times [-1, 2] \times [0, 3]. \]

- Of course, we can also integrate functions of three variables over more general regions. In keeping with the intuition that you built for double integrals, what we need to have is that one of the variables (let's just suppose it is \( z \)) should be bounded below and above by functions of the other two variables \( (u_1(x, y) \leq z \leq u_2(x, y)) \) and furthermore that one of the two other variables (let's say it is \( y \)) is bounded by functions of the remaining variable \( (h_1(x) \leq y \leq h_2(x)) \) and finally that the remaining variable is bounded by two finite constants \( a \leq x \leq b \). In such a case we would be able to compute the triple integral using an iterated integral of the form
\[ \int_{a}^{b} \int_{h_1(x)}^{h_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \, dy \, dx. \]

- I do not want to write out all of the possible forms this integral can take if we can write it in the different orders. The important thing to remember is that the innermost integral has endpoints that can depend on two variables, the middle integral can have endpoints that depend on one variable and the outermost integral has constant endpoints.

**Question 2:** Evaluate the triple integral of \( z \) over the solid tetrahedron bounded by the planes \( x = 0, y = 0, z = 0 \) and \( x + y + z = 1 \).

**Question 3:** Exchange the order of integration so that it is \( dy \, dx \, dz \):
\[ \int_{0}^{1} \int_{0}^{y} \int_{0}^{1} f(x, y, z) \, dz \, dx \, dy \]

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**Homework**

**Section 15.5:** Just browse through this section on your own to get an idea. No assigned problems.

**Section 15.6:** Read the section and do problems 3–11, 12*, 13–18, 29–30*, 33–34