Key Ideas: Double Integrals and Polar Coordinates

- Although it won't be mentioned until a later section of the book, I would like to introduce an important idea now because I think it will help you to understand some of the new material. It is important to recognize that coordinates are not “real”. Just like the different scales that we use to describe temperatures (Celsius, Farenheit and Kelvin) or the longitude and latitude lines on the the Earth, they are just things we made up. One useful technique throughout mathematics and its applications is change of variables. There's no reason to expect that $x$, $y$, and $z$ are the best ones to work through some problem. In fact, sometimes one has to change variables to get an answer, and sometimes one has to consider all possible changes of variables to see some important truth. We first will be looking at some special cases of alternative variable choices, but they will all fit together into a more general theory. You can change variables before evaluating an integral...but you'll have to add an extra Jacobian term to make it work.

- It is presumed that you have seen polar coordinates in a previous class. I will review the parts of it that will be most important to us in the context of this class, but you might want to look back at Section 10.3 for a more detailed discussion.

- **Polar Coordinates:** Let $(r, \theta)$ be a pair of real numbers. Viewing these as polar coordinates associates them to a point in the $xy$-plane by the formulas
  
  \[ x = r \cos(\theta) \quad y = r \sin(\theta). \]

  So, for every such pair there is a point in the plane. One can state this in words by saying that $(r, \theta)$ is the point on the circle of radius $r$ around the origin at a counter-clockwise angle of $\theta$ radians from the $x$-axis. However, there are lots of pairs that get sent to the same points. (For instance, notice that $(0, \theta)$ is always the origin regardless of $\theta$. Also, $(-r, \theta)$ gives the same point as $(r, \theta + \pi)$.)

- **Curves in Polar Coordinates:** In earlier classes, you have probably seen examples of curves plotted in polar coordinates. For example, consider the cardioid given by $r = 1 + \sin \theta$ or the four leafed rose $r = \cos 2\theta$ That are plotted on page 643.

**Question 1:** Plot the curve $r = \sin(2\theta)$.

- **“Rectangles” in Polar Coordinates:** Our main interest in polar coordinates will be their use in integrating over regions of the form

  \[ R = \{(r, \theta) | a \leq r \leq b, c \leq \theta \leq d\} \] .

  Note that this gives us the part of the region between the circles of radius $a$ and radius $b$ centered at the origin that are between the angles $c$ and $d$. So, in general, we will see regions like those shown on page 974.

**Question 2:** Express the disk of radius 2 centered at the origin as a polar rectangle. How about the region (“annulus”) between the circle of radius 1 and the circle of radius 3 centered at the origin?
• **What happens to $dA$ in Polar Coordinates:** We can take a polar rectangle $R$ and integrate any (continuous) function of $x$ and $y$ over it:

$$ f(x, y) \, dA. $$

It is clear how to rewrite $f$ in terms of $r$ and $\theta$ since $x = r \cos \theta$ and $y = r \sin \theta$. However, it is not immediately obvious how to rewrite $dA$!

In the case of the usual coordinates, we saw that $dA$ becomes either $dx \, dy$ or $dy \, dx$. This can be explained theoretically by noting that $dA$ is the area of a tiny rectangle with sides of length $dx$ and $dy$, so the simple product formula actually gives the relationship according to basic geometry. So, we need to consider what area a tiny polar rectangle has! Imagine a tiny rectangle where the angle changes by the amount $d\theta$ and the radius by $dr$...what is its area?

• Actually, that was a trick question, because you need more information. The area depends on $r$, since an angle of $d\theta$ makes a bigger difference for a bigger $r$! As you can see in figure 5 on page 976, the area of this rectangle is approximately $r \, dr \, d\theta$. (It is not exactly this, but in the limit as $dr$ and $d\theta$ get small, it gets more and more accurate.)

The $r$ that shows up there is the Jacobian of the transformation. Keep your eye out for other Jacobians as we continue through these topics over the next few days.

• **Double Integrals to Iterated Integrals:** This argument above motivates the following formula relating the double integral of a function over a polar rectangle to an iterated integral in terms of $r$ and $\theta$:

$$ \int\int_{R} f(x, y) \, dA = \int_{a}^{b} \int_{c}^{d} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta. $$

**Question 3:** Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

• Of course, one can consider more general regions as well. If $D$ is a region such that $r$ is constrained to be between two functions of $\theta$ then we have

$$ \int\int_{D} f(x, y) \, dA = \int_{h_1(\theta)}^{h_2(\theta)} \int_{c}^{d} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta. $$

**Question 4:** Find the area of one loop of the four leafed rose $r = \sin 2\theta$.

• (Hint for previous question: Note that $\int\int_{R} 1 \, dA = \text{Area}(R)!$)

**Homework**

**Section 10.3:** Look back at this section as necessary to remember how polar coordinates work.

**Section 15.4:** Read the section and do these problems: 1–4, 7,8*, 9–11, 12*, 15–22, 29–32