Key Ideas: Volume and Double Integrals

- We saw last time that if $f(x, y) \geq 0$ on a rectangle $R$, then the double integral of $f$ over $R$ is equal to the volume of the portion of space between the graph and the $xy$-plane. More generally, regardless of whether $f$ takes negative values, we can make the following statement:

**A General Geometric Interpretation for Double Integrals:** We can always say that a double integral is a difference of two volumes. Specifically, it is the volume of the region below the graph of $f$ and above the rectangle $R$ in the $xy$-plane minus the volume of the region below $R$ and above the graph:

$$\int\int_R f(x, y) \, dA = (\text{volume above}) - (\text{volume below}).$$

**Question 1:** Verify that $\int\int_R (x - 2y) \, dA = -0.5$ where $R = [0, 1] \times [0, 1]$. What does that mean?

**Answer:** The function $f(x, y) = x - 2y$ takes both positive and negative values on the square $[0, 1] \times [0, 1]$. If we compute the volume of the region of space that is above the $xy$-plane and below the graph of $z = f(x, y)$ we will get some number which I will call $V_1$. (I don't know what number it is.) Similarly, if we compute the volume of the part of space which is below the $xy$-plane and above the graph of $z = f(x, y)$ (which happens where $f$ takes negative values) we will get another number $V_2$. Both $V_1$ and $V_2$ are positive numbers. What we learn from the double-integral is that $V_1 - V_2 = -0.5$. Or, in other words: $V_2 = V_1 + 0.5$. A compact way to say it is this:

*The volume of the portion of space below the $xy$-plane and above the graph $z = x - 2y$ is exactly 0.5 units bigger than the volume of the portion of space above the $xy$-plane and below the graph of $z = x - 2y$."

Key Ideas: Double Integrals over More General Regions

- What is the volume of the “parabolic dome” under the graph of $f(x, y) = 4 - x^2 - y^2$ and above the $xy$-plane? I’m not sure, but I would like to be able to say that it is given by

$$\int\int_D 4 - x^2 - y^2 \, dA \quad D = \text{the unit disk centered at the origin.}$$

This would make sense, except for one problem. We haven’t defined what this means!
If you look back at our definition of the double integral, it really only works for rectangular regions. Now, we could (and some people do) come up with complicated alternative definitions that will work on any reasonable domain \( D \) in the plane, but that would be too much work. There is a really simple way to get around the problem, and this is what our book does.

**The Definition:** Let \( R \) be any rectangle that completely contains the bounded region \( D \) in the plane. Let

\[
F(x, y) = \begin{cases} 
  f(x, y) & \text{if } (x, y) \in D \\
  0 & \text{if } (x, y) \notin D
\end{cases}
\]

be a function which looks like \( f \) when in \( D \) but is zero everywhere else. Then we simply define

\[
\iint_D f(x, y) \, dA = \iint_R F(x, y) \, dA.
\]

Note that this has the same volume interpretation as for on a rectangle: the value of the double integral is always the volume of the part of space above the region \( R \) in the \( xy \)-plane and below the graph minus the volume of the part below \( R \) and above the graph. Theoretically, this solves our problem, because now we know what the integral of a function over any region means...but it does not tell us how to do it!

**More Practically:** We really can only integrate a function of two variables over two special types of regions in the plane. (We can then do better by starting with a more complicated region and breaking it up into little pieces of these two types as we'll see later.)

- **Type I:** A region \( D \) in the plane is said to be of Type I if it can be described as the region between two functions of \( x \) on an interval of \( x \) values:

\[
D = \{(x, y) : a \leq x \leq b, f_1(x) \leq y \leq f_2(x)\}.
\]

- **Type II:** A region \( D \) in the plane is said to be of Type II if it can be described as the region between two functions of \( y \) on an interval of \( y \) values:

\[
D = \{(x, y) : c \leq y \leq d, f_1(y) \leq x \leq f_2(y)\}.
\]

**Question 1:** Determine whether these are of Type I or II or both or neither: the top half of a unit disk, the space between two concentric circles, a triangle with one side lying the \( y \)-axis and another side lying on the line \( y = 2 \).

**Double Integrals become Iterated Integrals:** The reason we like regions of Type I and II is that double integrals become iterated integrals on these regions. In particular they become

\[
\int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y) \, dy \, dx \quad \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y) \, dx \, dy
\]

in the case of regions of types I and II respectively. If the region is both, then both of these work. If the region is neither then this idea does not work at all.

**Question 2:** Evaluate the integral of \( x + 2y \) over the region \( D \) which is bounded by \( y = 2x^2 \) and \( y = 1 + x^2 \).

**Question 3:** Find the volume of the solid that lies under the paraboloid \( z = x^2 + y^2 \) and above the region \( D \) bounded by \( y = 2x \) and \( y = x^2 \).
Homework

Section 15.3: Read the section and do these problems: 1–4, 7, 8*, 9–11, 21–23, 24*

Break: Because of Fall Break, we will not have class on Monday or Tuesday next week. Next Wednesday, there will be a guest lecturer in class and a group project. The homework from Sections 15.1 and 15.2 will be collected on Wednesday. The starred problems above will not be collected until Tuesday 10/20/09.