Evaluating Double Integrals as Iterated Integrals

- **Iterated Integrals:** When one or more partial integrals appear as the integrand (thing to be integrated) inside another partial integral, we call the result an *iterated integral*. In general, one must begin with the inner-most integral, evaluating them sequentially until all integrals have been complete. (Though, in the special case we will consider today where all of the endpoints are constant, the order is less important.)

**Question 1:** Evaluate \( \int_{0}^{2} \int_{0}^{1} x^2 y \, dy \, dx \).

- The main theorem that will allow us to evaluate double integrals is one that translates them into iterated integrals. Here is a hint of the proof of the theorem: Recall from your previous classes that you could find the volume of a solid running along the \( x \) axis from \( x = a \) to \( x = b \) if you knew the area \( A(x) \) of the slice at each given value of \( x \). As you may recall, the volume was then just \( \int_{a}^{b} A(x) \, dx \) because this “adds up” all of the areas to get the total volume. We can apply the same idea to the volume given by

\[
\int \int_{R} f(x, y) \, dA
\]

for a positive valued function on a rectangle \( R = [a, b] \times [c, d] \). Notice that at each slice for a given \( a \leq x \leq b \), the area is just \( A(x) = \int_{c}^{d} f(x, y) \, dy \). Then integrating this in \( x \) gives us:

- **Fubini’s Theorem:** If \( f \) is continuous on the rectangle \( R = [a, b] \times [c, d] \) then

\[
\int \int_{R} f(x, y) \, dA = \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy.
\]

**Question 2:** Verify that \( \int \int_{R} (x - 2y) \, dA = -0.5 \) where \( R = [0, 1] \times [0, 1] \).

- Wait, what does it mean when the integral is negative?

- We saw last time that if \( f(x, y) \geq 0 \) on a rectangle \( R \) then the double integral of \( f \) over \( R \) is equal to the *volume* of the portion of space between the graph and the \( xy \)-plane. More generally, regardless of whether \( f \) takes negative values, we can make the following statement:
A General Geometric Interpretation for Double Integrals: We can always say that a double integral is a *difference of two volumes*. Specifically, it is the volume of the region below the graph of $f$ and above the rectangle $R$ in the $xy$-plane **minus** the volume of the region below $R$ and above the graph:

$$\int\int_{R} f(x, y) \, dA = (\text{volume above}) - (\text{volume below}).$$

**Question 3:** Interpret the value of the previous double integral in terms of volumes associated with the graph of the function $f(x, y) = x - 2y$.

**Answer:** The function $f(x, y) = x - 2y$ takes both positive and negative values on the square $[0, 1] \times [0, 1]$. If we compute the volume of the region of space that is *above* the $xy$-plane and *below* the graph of $z = f(x, y)$ we will get some number which I will call $V_1$. (I don't know what number it is.) Similarly, if we compute the volume of the part of space which is *below* the $xy$-plane and *above* the graph of $z = f(x, y)$ (which happens where $f$ takes negative values) we will get another number $V_2$. Both $V_1$ and $V_2$ are positive numbers. What we learn from the double-integral is that $V_1 - V_2 = -0.5$. Or, in other words: $V_2 = V_1 + 0.5$. A compact way to say it is this:

*The volume of the portion of space below the $xy$-plane and above the graph $z = x - 2y$ is exactly 0.5 units bigger than the volume of the portion of space above the $xy$-plane and below the graph of $z = x - 2y$.*

**Question 4:** Evaluate the double integral

$$\int\int_{R} \frac{xy^2}{x^2 + 1} \, dA \quad R = [0, 1] \times [-3, 3]$$

using both orders of integration.

**Warnings:** Do not overestimate the power of Fubini's Theorem. It will not help you to evaluate the integral from Wednesday’s homework assignment, for example, because we do not know how to antidifferentiate the integrand. Also, do not get too used to the idea that you can just switch the order of integration easily and without changing the value. Next time, we will be dealing with the situation in which the inner integral has variable endpoints, and then switching the order of integration is a tricky ordeal!

**Homework**

**Section 15.2:** Read the section and do these problems: 1–15, 16*, 17–22