Key Ideas: Lagrange Multipliers

- Many of the max/min problems that we consider in this class as well as in calc 1 can be phrased in the following way: given a function \( f \) (of two or more variables), find the absolute extrema of \( f \) subject to the constraint \( g = k \) (for some other function \( g \)). For example:
  - “Find the maximum value of \( f(x, y, z) = x^2z + 2y^2 \) on the surface of the sphere \( x^2 + y^2 + z^2 = 1 \).” (This one is simply stated in the desired format).
  - “Find the absolute minimum value taken \( f(x, y) = xy - x^2 + 2y^2 \) on the closed, bounded region \( x^2 + 2y^2 \leq 4 \).” (Note that to answer this question, we would have to find the extreme values on the boundary as well as checking the critical points on the interior!)
  - “A train is constrained to travel on a track in the shape of the curve \( y = x^2 - 2 \) in the plane. If the temperature at the point \( (x, y) \) is given by \( f(x, y) = 1000/(x^2 + 2y^2 + 1) \), at what point on the track does the train encounter the highest temperature?”
    Here we want to maximize \( f \) subject to the constraint that \( g(x, y) = y - x^2 + 2 = 0 \).
  - “A farmer wants to fence a rectangular region of his field and divide it into two parts with a smaller fence down the middle. The fence around the outside costs $20/foot and the fence for the middle costs only $9/foot. What are the dimensions he should use to fence the largest area if he only has $1000 to spend?” Here we want to maximize the area \( f(x, y) = xy \) of the rectangle, but we know that the cost \( g(x, y) = 3x + 2y \) must satisfy \( g(x, y) = 1000 \). (We also need the \( x \) and \( y \) to be positive, but let’s ignore that criticism momentarily!)

- **The Idea:** As we know, if you are standing at a point and want to move in a direction to maximize the value of some function \( f(x, y) \) you would move in the direction that \( \nabla f \) points. But, what if you were not free to move in any direction. What if you were constrained to stay on the set of points that satisfy \( g(x, y) = k \) for some other function \( g \) and number \( k \). Then, you might not be able to go in the direction that the gradient points. But, unless the gradient of \( f \) was perpendicular to the curve, you could move in the direction closer to where the gradient points (the direction in which the angle made with the gradient is less than 90° rather than bigger). In fact, we can see from the definition of the directional derivative and the properties of the dot product that the function would increase in that direction. Using this procedure would lead you to a point where \( \nabla f \) is perpendicular to the curve...but at such a point \( \nabla f \) is parallel to \( \nabla g \)! This suggests we should look for places on the curve where the gradient of the function being optimized is a scalar multiple of the gradient of the constraint function.

- **Theorem:** For two differentiable functions \( f \) and \( g \), if \( f(x, y) \) has a maximum or minimum value on the smooth curve \( g(x, y) = k \) at the point \( (x_0, y_0) \) then there is some number \( \lambda \) such that:
  \[
  \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).
  \]
**Proof:** Let \( r(t) = (x(t), y(t), z(t)) \) be a parametrization of the curve \( g = k \). Then the question simply asks where the function \( f(t) = f(r(t)) \) has extrema. We know that \( f'(t_0) = 0 \) where ever it does, so suppose that \( r(t_0) = (x_0, y_0, z_0) \). Now we can use the chain rule to determine that \( f'(t_0) = \nabla f(x_0, y_0, z_0) \cdot r'(t_0) = 0 \). So, \( \nabla f \) (like \( \nabla g \)) is normal to the level curve at that point. We conclude that either they are non-zero and parallel, or one of the two is zero. Since the curve is smooth, we can choose \( r(t) \) so that \( r'(t) \) is never zero, and so the Lagrange multiplier equation as written must be satisfied (with \( \lambda = 0 \) if \( \nabla f \) is the zero vector).

- (The same idea applies for functions of three or more variables. If \( f(x, y, z) \) has a max or min on the level surface \( g = k \) then \( \nabla f \) and \( \nabla g \) are parallel 3-vectors.)

- **The method of Lagrange:** To find the maximum and minimum values of \( f(x) \) subject to the constraint \( g(x) = k \) (assuming these values exist!)
  1. Find all points \( x \) where \( \nabla f(x) = \lambda \nabla g(x) \) and \( g(x) = k \).
  2. Evaluate \( f \) at all of those points to find the largest and smallest values.

**Question 1:** Find the absolute maximum and minimum values that the function \( f(x, y) = 3x - 8y + 1 \) takes on the ellipse \( x^2 + 4y^2 = 25 \).
- Notice that we do not actually use the value of \( \lambda \) as part of our answer. In some more advanced applications of this method, the value of this Lagrange multiplier is important, but we will not see that in this class.
- If the closed bounded region on which one wants to find the absolute max/min values of a function takes the form \( g(x, y) \leq k \), then you can find the critical points on the interior and then instead of checking the boundary as we did last time, you can use Lagrange on the boundary (which is now \( g(x, y) = k \)).

**Question 2:** Find the extreme values of \( f(x, y) = x^2 + 2y^2 \) inside the disk \( x^2 + y^2 \leq 1 \).
- The procedure is the same (but the algebra more complicated) when there are more than two variables. In general, there are \( n + 1 \) equations involved in the Lagrange multiplier method when there are \( n \) variables.

**Question 3:** Use the method of Lagrange to find the points on the sphere \( x^2 + y^2 + z^2 = 4 \) that are closest to and farthest from the point \( (3, 1, -1) \).

**Homework**

**Section 14.8:** Read the section and do these problems: 3–11, 18–19