Finishing Up from Yesterday

1. I will show a movie of the tangent lines in different directions on a surface which may help you understand the notion of the directional derivative as a slope in a certain direction.

2. The gradient also is similarly useful in higher dimensions. For instance, if \( f(x, y, z) \) is a function defined at each point in space and we know the value at \((a, b, c)\), we might want to know what will happen to the value of the function if we move the input a little bit...but there are many different directions in which we can move. Just as in the two variable case (for nice functions \( f \)):
   
   - We can compute 
     \[
     D_uf(a, b, c) = \nabla f(a, b, c) \cdot u
     \]
     which is the directional derivative in the direction of the unit vector \( u \). (For example, if this is positive it means the function will increase if the input moves a little bit in that direction.)
   - The gradient vector \( \nabla f(a, b, c) \) points in the direction in which the value of \( f \) increases most rapidly from the point \((a, b, c)\) and its length is the rate of change in that direction.
   - If \( f(a, b, c) = k \) then the vector \( \nabla f(a, b, c) \) is orthogonal to the level surface \( f(x, y, z) = k \) and hence
     \[
     f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0
     \]
     is the tangent plane to the surface at that point.

3. (BTW: The word “nabla” for the symbol “\( \nabla \)” is Greek, not Hindi. Sorry ’bout that.)

Key Ideas: Maximum and Minimum Values

- You probably remember max/min problems from one-variable calculus. The idea there was to find the input value that makes the output of a function as big (or as small) as possible. There are many practical situations in which this is exactly what you want to do to optimize some situation to your benefit. Moreover, finding local maxima and minima, even if they were not the biggest or smallest values possible, were an important graphing tool helping you to identify points of special interest on the graph. All of this will be true in the multi-variable case as well.

- **Local and Absolute Extrema:** A function of two variables has a **local maximum** at \((a, b)\) if \( f(x, y) \leq f(a, b) \) for all points \((x, y)\) in some disk centered at \((a, b)\). Similarly, there is a **local minimum** at \((a, b)\) if \( f(x, y) \geq f(a, b) \) for all \((x, y)\) “near” \((a, b)\). If the inequalities hold not only in some little disk but in the **entire** domain of \( f \) then the function is said to be an **absolute maximum** (or minimum) at \((a, b)\).
• How did we find local extrema and global extrema in calc 1? We had to find critical points: places in the domain where $f' = 0$ or where $f'$ does not exist. Why? Because if neither of these is true, then $f' > 0$ or $f' < 0$ which could not possibly be a max or min since moving to the left or right will change the value. The same basic argument (augmented) works in this new situation as well.

• **Theorem:** If $f$ has a local extremum at $(a, b)$ and the first partial derivatives exist there, then $\nabla f(a, b) = \langle 0, 0 \rangle$. (So, you see, it is not the “derivative” but the gradient vector that now must be zero if it exists.) This can be proven by the same reasoning as before. Suppose that $(a, b)$ is a local maximum and that $\nabla f(a, b) = \langle 2, -1 \rangle$. This would mean that moving in the positive $x$-direction would increase the value of the function...so it can’t be a local maximum, can it? In fact, if either partial derivative was non-zero, we would know that $(a, b)$ is not a local extremum because $\nabla f$ always points in a direction of increasing and $-\nabla f$ always points in a direction of decreasing values.

**Question 1:** Super-simple example: The graph of $f(x, y) = x^2 + y^2 - 2x - 6y + 14$ is a paraboloid opening upwards. Use the theorem to find the location of its minimum.

• So, we use the word **critical point** to describe a point in the domain of $f$ at which $\nabla f = 0$ or $\nabla f$ does not exist. However, just as in the one variable case, you would be wrong to conclude that every critical point must be the location of a local extremum. These are “suspects”, but we have to prove that they are “guilty”!

**Question 2:** Find all critical points of $f(x, y) = y^2 - x^2$ and by figuring out what the graph looks like, determine whether they are really local extrema.

• One way to check whether you had an actual local extremum in the one-variable case was to use the second derivative test. It turns out that there is a higher dimensional version of this test that we can use here as well.

**Second Derivative Test:** Suppose that the second partial derivatives of $f$ are continuous on a disk centered at the critical point $(a, b)$. Then let

$$D = \frac{f_{xx}(a, b)}{f_{yy}(a, b)} - \left(\frac{f_{xy}(a, b)}{f_{yy}(a, b)}\right)^2 = \frac{f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2}{f_{yy}(a, b)}.$$

- If $D > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum.
- If $D > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum.
- If $D < 0$ then $f(a, b)$ is not a local extremum.
- If $D = 0$ or $f_{xx} = 0$ then the test provides no useful information – you have to figure it out some other way.

**Question 3:** (Example 3, Page 924) Find the local extrema and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$. 
• **How to solve** $\nabla f = 0$: When you try to find where the gradient vector is equal to zero, you will have two equations to solve simultaneously. In the super-easy example above, each could be solved separately since one was in terms of $x$ and the other in terms of $y$. But, if they have variables in common it is not so simple. Two approaches that generally work are to (a) add some multiple of one equation to the other to eliminate some terms or (b) use one equation to determine a formula for one variable in terms of the other and substitute into the second equation. Sometimes, the second method requires considering several different possibilities, each of which must be handled separately.

**Question 4:** Find the local extrema and saddle points of $f(x, y) = 100xy - x^2y - y^2x$.

• **What can go wrong?** There are many equations (most, in fact) that we cannot solve exactly. In example 4 on page 925, the locations of the critical points are only estimated using a calculator’s “solve” routine. Another thing that could go wrong is that there could be infinitely many critical points, or that they could be points of non-differentiability in which case the test does not apply. For example, the function $f(x, y) = |x| + |y|$ is a perfectly reasonable function with an absolute minimum at $(0, 0)$, but critical points everywhere on both axes since the partial derivatives fail to exist.

• Of course, there are also “word problems” in which a situation is described and part of your job is to figure out which function it is you want to optimize. For example:

**Question 5:** A company sells two products which are partial substitutes for each other, such as coffee and tea. If the price of one product rises, then demand for the other product rises. The quantities demanded, $q_1$ and $q_2$, are given as functions of the prices, $p_1$ and $p_2$ by

$$q_1 = 517 - 3.5p_1 + .8p_2 \quad q_2 = 770 - 4.4p_2 + 1.4p_1.$$ 

What prices should the company charge in order to maximize the total sales revenue? (Hint: The revenue for each item will be the product of the price and the quantity sold.)

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**Homework**

**Section 14.7:** Read the section (we did not cover the end yet) and do these problems: 1, 2, 5–9, 10, 11, 12, 15, 16

**Note:** I don’t believe it, but next week is already our second test, so no problems will be collected and graded. Consider #10 as a really good one to work on. Can you find and classify all four critical points?