Key Ideas: Some Chain Rules / Implicit Differentiation Done Right

- Today we will learn the general chain rule for functions of more than one variable, along with some special cases that work only for certain numbers of variables. We will use this rule to finally get a simple formula for the result of implicit differentiation.

- Suppose the function \( f(x, y) \) gives some information about what is happening at the point \((x, y)\) on the plane. For example, suppose it gives the temperature at that point. Now, remember what it means to have a vector valued function \( r(t) = \langle x(t), y(t) \rangle \) that gives you a point in the plane for each value of \( t \). We can think of this as a moving particle whose position at time \( t \) is given by \( \langle x(t), y(t) \rangle \). Putting these two ideas together, we see that \( f(x(t), y(t)) \) gives the temperature that the particle is experiencing at time \( t \). The simplest version of the chain rule that we will learn will allow us to answer this question: at what rate is the temperature experienced by the particle changing at time \( t \)? In other words, what is \( \frac{df}{dt} \)?

- Let’s continue with this same example and be more specific so that we can see precisely how the numbers work. Suppose \( r(0) = \langle 0, 0 \rangle \) (starts at origin). Suppose also that we know \( f(0, 0) = 10 \). Further suppose that \( f_x(0, 0) = 2 \) and \( f_y(0, 0) = -3 \) (increasing \( x \) increases \( f \) by twice the same amount and increasing \( y \) decreases \( f \) by three times that amount). How would we know what happens to the temperature? We would have to know which direction the particle is moving in, right? I mean, if only its \( x \)-coordinate is increasing, we would expect the temperature to increase and if only its \( y \)-coordinate is increasing we would expect the temperature to decrease. More generally, this should somehow depend upon \( r'(t) = \langle dx/dt, dy/dt \rangle \). Then, it is not hard to see that

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \langle f_x, f_y \rangle \cdot \langle x', y' \rangle.
\]

This is the Chain Rule (case 1). (Notice how writing it as a dot product of two vectors makes it look more like the usual chain rule. We will learn more about that on Monday.)

Unlike Calc 1 where the chain rule was a matter of daily survival (how often have you needed to know it to take a derivative?!?) the Chain Rule for functions of more than one variable is more a theoretical tool for making discoveries and proving theorems. When it comes to the “grunt work” of actually computing a derivative, it is generally possible to do it without making use of these rules. In scientific applications, it is sometime used when the values of some functions and derivatives are known experimentally but not in terms of a formula, as in this example:

**Question 1:** The strength of an electromagnet \( S(T, V) \) is a function of the temperature and the voltage. We have determined experimentally that \( S(120, 80) = 480, S_T(120, 80) = \)
−.4, and \( S_V(120, 80) = 2 \). If the temperature is 120° and increasing at 2° per hour \( (dT/dt = 2) \) and the voltage 80 but decreasing at 3 volts per hour \( (dV/dt = -3) \), what is \( dS/dt \) and what does that tell us?

• Now suppose that \( x \) and \( y \) depend not only on \( t \), but also on another variable \( s \)! So, we have the two variables as functions of two variables. We can then consider the partial derivatives \( \partial f/\partial s \) and \( \partial f/\partial t \). How would we find these? Well, as we know, to find either one you just pretend the other variable is constant and so we end up with the same formula as above, but written all with partial’s. **The Chain Rule (Case 2):**

\[
\begin{align*}
\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}, \\
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}.
\end{align*}
\]

• The most general situation of the chain rule is for a function \( f(x_1, x_2, \ldots, x_n) \) of \( n \) variables where you also view each of the \( x_i \)'s as a function of \( m \) variables \( t_1, \ldots, t_m \). You can only (partially) differentiate this with respect to one of the variables at a time and for each \( t_i \) you get

\[
\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial t_i}.
\]

• **Recall Implicit Differentiation:** What is the point of *implicit differentiation*? This is where you find the slope on the graph of an equation which is *not* the graph of a function. (If it were a function’s graph, you could use just regular differentiation!) For instance, the graph of \( x^4 + y^4 = 1 \) looks sort of like a squashed circle. If I have a point \((x, y)\) on this geometric object, how can I find the slope of the tangent line there? We learned a way to do it in Calc 1…but today we’ll learn another method that is sweeter!

• Moreover, we can ask the same question about the graph of an equation in three or more variables. **What is the equation of the tangent plane to the graph \( x^4 + y^4 + z^4 = 1 \) at some point on this surface?** We could have read about an algorithm for finding the appropriate derivatives on page 883…but I skipped that because I knew that the method we are learning today is actually better.

• **Application of the Chain Rule to Implicit Differentiation – Calc 1 Case:** You recall that in calc 1, “implicit differentiation” meant finding \( \frac{dy}{dx} \) when you knew an equation like \( x^2y^4 - 2y = 0 \). As we learned it, implicit differentiation was a *procedure* that had to be worked through, but there was no formula for \( dy/dx \) that always worked. Now, we will see that there is such a formula. It can be written in terms of partial derivatives and derived using the chain rule that we learned last time.

Consider the graph of the equation \( F(x, y) = 0 \). (This graph is a curve in the plane.) We want to know the slope of the curve \( (dy/dx) \) at each point on the curve. The trick that
makes implicit differentiation work is the assumption that \( y = y(x) \) is a function of \( x \). Then, the chain rule (case 1) gives us

\[
\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0.
\]

Using the fact that \( \frac{dx}{dx} = 1 \) and solving for \( \frac{dy}{dx} \) we get

\[
\frac{dy}{dx} = -\frac{F_x}{F_y}.
\]

**Question 2:** Find the slope of the graph of \( xy + x = \sin(xy) + 2y \) at \( (0, 0) \) in two different ways: first as you would have done it in Calc 1 and then using this new formula.

- **Application of the Chain Rule to Implicit Differentiation – Multivariable Case:**
  Similarly, the same procedure can be used to derive a general formula for the partial derivatives to a surface defined by an equation \( F(x, y, z) = 0 \). It turns out to be just

\[
\frac{\partial z}{\partial x} = \frac{F_x}{F_z} \quad \frac{\partial z}{\partial y} = \frac{F_y}{F_z}.
\]

- If the point \( (a, b, c) \) is on the surface \( F(x, y, z) = 0 \) and if its tangent plane at that point is not vertical, you can write the tangent plane as

\[
z = \frac{\partial z}{\partial x}(a, b, c)(x - a) + \frac{\partial z}{\partial y}(a, b, c)(y - b) + c.
\]

**Question 3:** Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) for the surface \( x^3 + y^3 + z^3 + 6xyz = 2 \). What is the tangent plane at \( (1, 0, 1) \)?

**Homework**

**Section 14.5:** Read the section and do these problems: 13–16, 27, 28*, 29–35, 36*