Quick Thinking: How fast can you find these derivatives?

1. \( u(x, t) = \sin(xt^2) \quad u_{xt}(x, t) = \)

2. \( f(x, y, z) = x^2yz + \frac{z^2}{y^2 + 1} \quad f_{zx}(x, y, z) = \)

Key Ideas: Limits and Differentiability

• As in Calculus I at this college, we will not give the ideas of “limits” and “continuity” their deserved coverage in this class. These are really important ideas. If calculus was a car, these would be the engine and the fuel. Since we will not be learning enough about them, it is as if we are learning to drive the car here but not how to build it from scratch. I will tell you just a little bit about these ideas (enough to understand how the car works or maybe even to fix a minor breakdown) but for more details you will have to read the section yourself or take a course in analysis.

• For a function \( f(x, y) \) of two variables and fixed real numbers \( x, y \) and \( L \), the statement

\[
\lim_{(x,y) \to (a,b)} f(x, y) = L
\]

means loosely that the output of the function \( f \) “approaches” the value \( L \) as the input values get closer to \((a, b)\).

• More specifically, it means that you can ask me to give you a little circle centered at the point \((a, b)\) for which the first \( n \) digits of \( L \) and \( f(x, y) \) agree if I take \((x, y)\) to be any point in that circle (other than the center). If you ask me to get the first 100 digits to agree, perhaps I could say that a circle of radius \( .4 \) has the property that all of the output values look like \( L \) to the first 100 digits. If you ask for more digits (like 2 million) I would probably have to take a smaller circle. But, the claim is that I could get as many digits as you ask for, as long as I make the circle small enough.

**Question 1:** Estimate \( \lim_{(x,y) \to (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} \) and \( \lim_{(x,y) \to (0,0)} \frac{3x^2y}{x^2 + y^2} \) by plugging in values of \((x, y)\) near \((0, 0)\).

**Question 2:** In contrast try values of \( \frac{xy}{x^2+y^2} \) near \((0, 0)\) but either on the line \( y = 0 \) or on the line \( y = x \).

• **A way to show the limit does NOT exist:** If the value of \( f \) approaches different values when \((x, y)\) approaches \((a, b)\) along different paths, then the limit does not exist.

• Then, “continuous” means that the above equation and \( f(a, b) = L \) are both true. (So, the value the function approaches is equal to the value it has at that point.)
We saw last time that the tangent plane can be used to approximate the values of \( f(x, y) \) near the point of tangency. But, does that always work? Sometimes, \( f_x \) and \( f_y \) are not defined at all...and sometimes they are defined but the graph still does not look like its tangent plane!

The function \( f(x, y) = |x| + |y| \) is an example for which the partial derivatives are not defined at all along the lines \( x = 0 \) and \( y = 0 \) in the \( xy \)-plane because the graph there is “sharp”.

The function \( f(x, y) = \frac{xy}{x^2 + y^2} \) gives a good example of a function that does not look like its tangent plane even though \( f_x \) and \( f_y \) are defined if we just add the definition \( f(0, 0) = 0 \) so that it is defined everywhere. Note that \( f_x(0, 0) = 0 \) and \( f_y(0, 0) = 0 \), but it does not look like the tangent plane \( z = 0 \) if you are on the line \( y = x \)!

**Definition:** We want to rule out all of the “bad places” on the graph of a function at which the graph does not look locally like a plane. Therefore, unlike the one-dimensional case, we say by definition that a function \( f(x, y) \) is differentiable at \((a, b)\) if it looks like the tangent plane at that point. (This is a loose way of saying the same thing as theorem 7 on page 911.)

**How to tell if it is differentiable?:** This criterion (which is not equivalent to the definition, it is weaker) is easier to use than the real definition: If \( f_x \) and \( f_y \) are continuous at \((a, b)\) then \( f \) is differentiable at \((a, b)\).

If a function is differentiable, then you can use the tangent plane as a linear approximation to the function itself.

**Question 3:** Show that \( f(x, y) = xe^{xy} \) is differentiable at \((1, 0)\). This justifies our use of linear approximation to estimate \( f(1.03, -0.01) \) yesterday in class!

Although we lose the visual aspect of the geometry, everything we’ve said continues to apply in the higher dimensional cases as well. For example, for a function of 3-variables we can say that \( f \) is differentiable at \((a, b, c)\) if

\[
f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x-a) + f_y(a, b, c)(y-b) + f_z(a, b, c)(z-c).
\]

**Homework**

**No HW Tonight:** On Friday we’ll be seeing the Chain Rule and Implicit Differentiation.