Practicing Contour Plots

**Question 1:** Plot the contours of \( f(x, y) = \frac{1}{x^2 + y^2} \) at heights \( z = 1, 4, 9 \).

**Question 2:** What is the level curve of \( z = y - x^2 \) at height \( z = k \)? (Note: This is the same as the question “What is the trace on the plane \( z = k \)?”) Plot a contour map showing the contours at heights \( z = -2, 0, 2 \).

**Key Ideas: Partial Differentiation**

- What was the role of the derivative in 1-variable calculus? From the point of view of functions, it gave us an indication of whether a function was increasing or decreasing. Moreover, it told us by how much the output of the function would increase or decrease for a small change in \( x \).

  For example, if \( f(x) = 1 - x^2 \), then the fact that \( f'(1) = -2 \) means that \( f \) is decreasing at \( x = 1 \). In fact, a tiny increase of \( x \) from \( x = 1 \) would lead to a decrease in \( f \) by about twice that much. In contrast, if \( g(x) = x^2 + 1 \) then the fact that \( g'(1) = 2 \) means that increasing \( x \) from 1 in this case will increase \( f \) by about twice the same amount.

- What could play the same role in the case of functions of two variables? It is not so simple to say that a function is “increasing” or “decreasing” anymore. Remember the “saddle” shaped surface \( z = x^2 - y^2 \)...whether you increase or decrease depends upon which variable you choose to increase.

- Consider the function \( f(x, z) = x^2 - y^2 \). The point \((1, 1, 0)\) is on this graph. In the plane \( x = 1 \) we see it as part of the downward parabola \( z = 1 - y^2 \). Using ordinary differentiation, we can see that the slope there is \(-2\), meaning that increasing \( y \) by a tiny amount leads to a decrease in \( z \) by about twice that much. Similarly, we can look at this same point on the trace of the surface in the plane \( y = 1 \) so that we see the curve \( z = x^2 - 1 \). Now, the slope of this curve at the point is \( 2 \), meaning that an increase in \( x \) will lead to an increase of \( z \) by twice as much.

- As it turns out, these two pieces of information (the “derivatives in the \( x \) and \( y \) directions”) are all we need to know in order to generalize the most important results of calculus to this multivariable case.

**Partial Derivatives:** For a function \( f(x, y) \) we define

\[
\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}
\]

\[
\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}
\]

- In practice, all this means is that you treat one variable as a constant and differentiate with respect to the other. So, for example, with \( f(x, y) = x^2 y + x - y \) we have \( f_x(x, y) = 2xy + 1 \) and \( f_y(x, y) = x^2 - 1 \). That’s it! There are no new tricks to learn for differentiation apart from remembering how to treat a parameter as a constant. (Of course, just as before, not every function comes with a formula or is made out of the functions whose derivatives you have memorized, so there is always the possibility that you will have to resort to using the definition of the derivative in order to find it, but that will not arise often in this class.)
• For a geometric interpretation of the values of this function, you can try to imagine the graph of \( f(x, y) \) with two lines passing through the point \((a, b)\): a line parallel to the \(xz\)-plane with slope given by \(f_x(a, b)\) and a line parallel to the \(yz\)-plane with slope given by \(f_y(a, b)\). Both of these lines will be tangent to the surface (since they are tangent to the curves in the trace), and you can imagine the plane containing them as the tangent plane...something we will study in more detail later.

• **Higher Derivatives:** Since the derivatives \(f_x\) and \(f_y\) are functions, there is no reason that you can’t also take their derivatives. Then you find

\[
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right).
\]

But, moreover, you now also have the mixed partial derivatives which involve some differentiation in each variable:

\[
\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right).
\]

**Question 1:** Compute the mixed partial derivatives of \( f(x, y) = x^2 y + x - y \).

• Notice that the mixed partial derivatives are equal. The good news is that this is usually the case. For all nice functions, the mixed partial derivatives are equal. **Clairaut’s Theorem** (page 885) makes this explicit by stating that the mixed partials are equal in any disk where they are both continuous. (In other words, the only possibility of them being unequal will occur when one of them is undefined at some point.) A trivial example is the function \( f(x, y) = x^{2/3} \) since \( f_y = 0 \) and hence \( f_{yx} = 0 \), but \( f_x \) is undefined at \( x = 0 \) and hence \( f_{xy} \) is too!

• One of the most useful (and, at least to me, most interesting) areas of mathematics is the study and solution of partial differential equations. For instance, the equation

\[
f_{xx} = cf_{tt}
\]

describes the motion of a vibrating guitar string. (This can be derived by studying how the neighboring parts of the string affect each other.) Here \( x \) is to be thought of as the position on the string and \( t \) as “time”. Although this equation was originally studied and solved simply for the apparently useless goal of understanding why guitar strings vibrate, it lead to the discovery of radio waves (when the same equation showed up in the relationship between electrons and magnets) and hence to the invention of many modern conveniences: radio, TV, microwaves, CD players, etc.

Important Comment: When we write a differential equation like \( f_x + (f)(f_y) = 0 \) we do not mean that we should find values of \( x \) and \( y \) that make the equation true, but rather that it is true for all values of \( x \) and \( y \). In other words, the functions on either side of the equal sign should be the same. For example, \( f(x, y) = x^2 + y^2 \) satisfies \( 4f = (f_x)^2 + (f_y)^2 \) because both sides of the equation are \( 4x^2 + 4y^2 \). But, it does not satisfy \( 4f = (f_x)^2 - (f_y)^2 \)....even though the two sides are equal when \( y = 0 \).

• There are also partial derivatives for functions of more variables, with the obvious notations and definitions. In those cases, the derivative indicates how much the function will change as each variable is increased by tiny amounts, but it is harder to picture the geometric implications.

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**Homework**

**Section 14.3:** Read the section carefully and do these problems: 1, 5–8, 11, 12, 15–31, 39, 40*, 51–53, 54*, 61–65, 71, 72*, 73