Key Ideas: Functions of Two Variables

- **Definition:** A function of two variables \( f(x, y) \) assigns a real number to certain ordered pair of real numbers. The set of all such pairs for which \( f(x, y) \) gives such a value is called the **domain** \( D \) of the function \( f \) and can be regarded as a subset of the plane. Finally, the **range** of \( f \) is the set of all the values that the function takes on its domain: \( \{ f(x, y) \mid (x, y) \in D \} \). (The range is a subset of the number line, not the plane!)

- In determining the domain of these functions, it may be helpful for you to recall that the functions below are only defined for the specified values:

  \[
  \begin{align*}
  \ln(x) & \text{ only for } x > 0 \\
  \sqrt{x} & \text{ only for } x \geq 0 \\
  \frac{c}{x} & \text{ only for } x \neq 0
  \end{align*}
  \]

- Functions of two variables can sometimes be written in terms of formulas involving \( x \) and \( y \). For example:

  \[
  f(x, y) = x^2 - y^2 \quad g(x, y) = e^{x+y} \quad h(x, y) = \sqrt{4 - x^2 - y^2}
  \]

**Question 1:** Identify the value at the point \( (2, 1) \), the domain and range of each of the functions described above.

- Some functions of real interest cannot be written in terms of formulas, but are nevertheless functions that we will want to understand and study using the techniques developed in this class. Consider, for example, the function which gives the temperature at a given position on a rectangular piece of metal whose corner is being heated, or the height above sea level at a point on the Earth’s surface specified by longitude and latitude, or the number of cans of Coke that will be sold if it costs \( x \) cents per can and Pepsi costs \( y \) cents per can!

- Just like functions of one variable (recall tables that give the temperature as a function of time from calc 1), functions of two variables that cannot be described by a formula can be at least partially described by a table of values. This table now has to have a cell giving the value for a given value of each variable, as specified by a particular column and row. (See example on page 856.)

- The **graph** of a function of two variables is the graph of the equation \( z = f(x, y) \).

  Note that this is a special case of the **surfaces** we saw before. (Only those surfaces whose equations can be “solved for \( z \)” are graphs of functions of \( x \) and \( y \).)
• The tricks for figuring out what the graphs look like are simply the same as we used in the previous chapter for figuring out what the graphs of arbitrary equations look like. One of the key tools, the method of traces, takes a slightly different form as the level curves of a function which are the curves \( f(x, y) = k \) for different values of \( k \) in the range of \( f \). You can then plot the level curves together on a single set of axes producing a contour map. (See page 860–864). As a general guideline, it is helpful to remember that when level curves get close on a contour map, this indicates that the graph is steep, while level curves far apart indicate a lot of “flatness”.

Question 2: Sketch the contour map of each of the functions above by plotting level curves for \( k = -2, -1, 0, 1, 2 \).

• Though much of our geometric wisdom falls apart, we often also want to consider functions of three or more variables. For instance, we might want to know how much our profit will be as a function of our production level of five different products, or the temperature at each point in a 3-dimensional room. For the most part, everything is the same in these cases except that we lose the ability to picture the graphs in any realistic way. The trick of “level curves” still works for functions of three variables, but we end up instead with level surfaces. For instance, suppose the origin is at the very center of a room whose temperature at each point in space is given by the function \( T(x, y, z) = x^2 + y^2 + z^2 \). Then the level surfaces will be spheres around the center of the room on which the temperature has a given value with the temperature increasing rapidly as you move away from the origin. (I suppose there must be something very cold right there.)

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**Homework**

**Section 14.1:** Read the section and do problems 5–10, 21–26, 30, 32, 33, 40, 41, 42, 43–46

**IMPORTANT:** There are no problems to be collected next Tuesday because we have agreed to have another First Test. (So many students scored poorly on the first one that we are trying again. The higher of the two scores will be retained. The test will not have the same questions, but will be about the same length and difficulty. This time, one of the questions will be about partial derivatives!)