Key Ideas: Curvature and Arclength

- **Key Points From Last Time:** The formula
  \[ s(t) = \int_a^t |r'(u)| \, du \]
gives the “arclength function”. The interpretation is that this function measures the length of the space curve described by \( r(t) \) from the point \( r(a) \) to \( r(t) \) (unless the curve happens to repeat itself, in which case the value you get is too big.)
  - Sometimes, you can actually compute this exactly. In which case you can really find the length of some part of the space curve or the arclength function.
  - Sometimes, you cannot perform the integration exactly because you do not know an anti-derivative for the integrand. In those cases, you can still approximate the value of the integral by using some numerical methods (such as Simpson’s rule or the integration routine in your calculator).

- **New Definition:** We say a vector valued function \( r \) is an arclength parametrization of its space curve if \( |r'| \equiv 1 \) (i.e. if the speed is always equal to one). Usually, we use the parameter \( s \) instead of the parameter \( t \) for an arclength parametrization because if \( |r'| \equiv 1 \) then the length of the arc from \( r(a) \) to \( r(b) \) is always equal to \( |b - a| \).

- **Turning Any Parametrization into an Arclength Parametrization:** Find the arclength function \( s(t) \), solve for the inverse function \( t(s) \) and substitute it into the original function \( r \) to get \( r(s) \), which is an arclength parametrization of the same spacecurve.

**Question 1:** Find the arclength parametrization of \( r(t) = \langle \cos(t), \sin(t), t \rangle \).

- Now, anytime you are given \( r(t) \) we can also (implicitly) talk about its dependence on the arclength parameter \( s \) as well as \( t \). In today’s discussion is will be useful for us to jump back and forth on occasion.

- Whenever \( r \) is a smooth parametrization, we can talk about \( T = r'/|r'| \) which we call the “unit tangent vector”. Well, today’s topics both grow out of the importance of its derivative \( T' \). We want to notice a number of important things:

  - Note that computing \( T' \) is much harder than, but not at all the same as, computing \( r'' \).

**Question 2:** For \( r = \langle 3t^2, 3t, 2t^3 \rangle \), find \( r', r'', T \) and \( T' \). Note that although \( r' \) and \( T \) are almost the same thing (in fact, they are parallel),

\[
    r'' = \langle 6, 0, 12t \rangle \quad \text{and} \quad T' = \left\langle \frac{2 - 4t^2}{(1 + 2t^2)^2}, \frac{-4t}{(1 + 2t^2)^2}, \frac{4t}{(1 + 2t^2)^2} \right\rangle
\]

are very different. Still, it is \( T' \) that will be important in today’s lecture.
No matter what \( r \) is, \( T \) and \( T' \) are orthogonal! (This may seem surprising, but it is easy to prove by differentiating the equation \( T \cdot T = 1 \).

In fact, \( T' \) points in the direction that the curve is turning or bending just as \( T \) points in the direction it is going. (For instance, if \( r \) describes a straight line, the \( T' \) is the zero vector!)

The length of \( T' \) has something to do with how much it is turning.

This motivates us to make some definitions. The first is this: To go along with the unit tangent vector \( T \), we define \( N = \frac{T'}{|T'|} \) (the unit normal vector to a space curve).

**Question 3:** Find \( T \) and \( N \) for the space curve above at the point \((3, 3, 2)\). (Note that \( T' = \langle -2/9, -4/9, 4/9 \rangle \) and \( |T'| = 2/3 \).)

**Curvature** is a number which measures how curvy something is. We would like to say that something flat (like a line) has curvature \( \kappa = 0 \), and that the more curvy a geometric object is at a point the more positive the curvature will be. Surprisingly, it turns out that curvature is not only useful in abstract geometry, but is of fundamental importance in physics as well. Gravity turns out to be nothing other than the curvature of space, and even wave dynamics depends on expressing things in terms of curvature. In this class, we will only be interested in curvature of space curves, but this will turn out to be related to the physics of an object moving along a path.

But, how do we actually define “curvature”? It might make sense to think of the length of \( T' \) as being the curvature...but that depends upon the choice of parametrization and we’d like it to be something about the space curve itself. We come up with two different (but equivalent) definitions that build on this idea but do it in a way that depends only on the curve.

**Formulas for Curvature:** We denote the curvature by \( k \):

\[
\begin{align*}
  k(s) & = \left| \frac{dT}{ds} \right| = |r''(s)|, \\
  k(t) & = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}
\end{align*}
\]

The first of these two formulas is the one that “makes sense”. We just define the curvature to be the length of \( T' \) for the arclength parametrization. However, it is difficult (and sometimes impossible) to find the arclength parametrization, so we make use of the lemmas and the chain rule to rewrite the formula in terms of \( r(t) \) (the second formula).

I will not hold you responsible for the proof of the second formula, but if you are interested, here is a quick summary. First, it follows from the FTC that \( s'(t) = |r'(t)| \). Then, from the chain rule (and the formula we just found) we get

\[
\frac{dT}{dt} = \frac{dT}{ds} \frac{ds}{dt} \Rightarrow |T'(t)| = k|r'(t)|.
\]

Moreover, \( |T'(t)||r'(t)|^2 = |r'(t) \times r''(t)| \) (see page 833 for details, but this is essentially due to the distributive property of the cross product and the fact that the length of the cross-product of orthogonal vectors is just the product of their lengths). Using all of this we can replace the arclength dependent definition of \( k \) by another definition with no \( s \).

**Homework**

**Section 13.3:** Read the rest of the section and do these problems: 13–25

**Worked Example:** Look at the next page for a thoroughly worked example on arclength parametrization and curvature.
A Worked Example for Curvature and Arclength Parametrization

If \( \mathbf{r}(t) = (e^t \sin(t), e^t \cos(t)) \)
then by differentiating we find that
\[
\mathbf{r}'(t) = (e^t (\cos(t) + \sin(t)), e^t (\cos(t) - \sin(t))).
\]

Now finding the length of this vector we have
\[
|\mathbf{r}'(t)| = \sqrt{e^{2t} (\cos^2 t + 2 \cos t \sin t + \sin^2 t) + e^{2t} (\cos^2 t - 2 \cos t \sin t + \sin^2 t)} = \sqrt{2}e^t.
\]

We can integrate this vector length to find the arclength of the curve from \( t = 0 \) to an arbitrary value of \( t \):
\[
s = \int_0^t \sqrt{2}e^u \, du = \sqrt{2}(e^t - 1).
\]

Then, solving for \( t \) in terms of \( s \) gives: \( t = \ln \left( \frac{s}{\sqrt{2}} + 1 \right) \). Resubstitute back into the original function to get the arclength parametrization
\[
\mathbf{r}(s) = \left( \left( \frac{s}{\sqrt{2}} + 1 \right) \sin \left( \ln(s/\sqrt{2} + 1) \right) \right), \left( \frac{s}{\sqrt{2}} + 1 \right) \cos \left( \ln(s/\sqrt{2} + 1) \right) \right).
\]
This really does have the property that
\[
\mathbf{r}'(s) = \frac{1}{\sqrt{2}} (\cos(\zeta) + \sin(\zeta), \cos(\zeta) - \sin(\zeta)) \quad \zeta = \ln(s/\sqrt{2} + 1)
\]
is a unit vector.

Since we have an arclength parametrization, we can find the curvature as \( |\mathbf{r}''(s)| \):
\[
\kappa_1(s) = |\mathbf{r}''(s)| = \frac{\sqrt{2}}{2 + \sqrt{2} s}
\]
or we can use the formula in terms of \( t \) and find it as
\[
\kappa_2(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{1}{\sqrt{2}e^t}.
\]
(Note that I found the cross product by adding a zero as a third component on \( \mathbf{r}' \) and \( \mathbf{r}'' \) to make them 3-dimensional.)

Now, suppose we want to find the curvature of this curve at the points \( (0, 1) \) and \( (0, e^{2\pi}) \). Note that we can think of this as being when \( t = 0 \) and \( t = 2\pi \) or as being when \( s = 0 \) and when \( s = \sqrt{2}(e^{2\pi} - 1) \). We can use either in the appropriate formula to get the curvature.

Thus, the curvature at \( (0, 1) \) is \( \kappa_1(0) = 1/\sqrt{2} \) and \( \kappa_2(0) = 1/\sqrt{2} \). More interestingly, the curvature at \( (0, e^{2\pi}) \) is \( \kappa_1(\sqrt{2}(e^{2\pi} - 1)) = \frac{\sqrt{2}}{2 + 2(e^{2\pi} - 1)} \) and \( \kappa_2(2\pi) = \frac{1}{\sqrt{2}e^{2\pi}} \). It is not a coincidence that these are two different ways to write the same number. They are equal because they are both the curvature of the same curve at the same point, just found in two different ways. (The formulas worked out a bit nicer here when using \( t \) than when using \( s \)...but in many cases the formulas with \( s \) are literally impossible and hence the \( t \) formula is the only way to go. Still, the \( s \) formula is the one that makes sense when you look at it and that's why it is useful in theoretical situations.)