Key Ideas: Vector Valued Functions and Tangent Vectors

- Some of today’s material (the physical interpretation, in particular) actually does not get covered until section 13.4, but I think it is easier to mention it now. So, please forgive me for covering it “out of order”.
- Differentiation of vector functions is done componentwise

\[ \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle. \]

And, of course, higher derivatives are just the same procedure applied repeatedly. So, for instance:

\[ \mathbf{r}(t) = \langle e^{2t}, 2t, 5 \rangle \Rightarrow \mathbf{r}'(t) = \langle 2e^{2t}, 2, 0 \rangle \Rightarrow \mathbf{r}''(t) = \langle 4e^{2t}, 0, 0 \rangle \Rightarrow \cdots \]

You will find that all of the familiar rules of differentiation from single variable calculus continue to be true in the higher dimensional context. (See page 826.) However, some of them take on new meanings now.

**Question 1:** Use the “product rule” \( \frac{d}{dt}(u \cdot v) = u' \cdot v + u \cdot v' \) to prove that if \( |\mathbf{r}(t)| = c \) then \( \mathbf{r}' \) is orthogonal to \( \mathbf{r} \).

- When we draw the “tangent vector” \( \mathbf{r}'(t) \) we usually put it at the point of \( \mathbf{r} \)...that is, it’s tail touches the curve at \( \mathbf{r}(t) \).
- The book gives a more involved definition for differentiation, one that is reminiscent of the calc 1 definition with limits. Let’s think about what it would mean. Since we know that if \( \mathbf{r}(t) = \langle f(t), g(t) \rangle \) then

\[ \mathbf{r}'(t) = \langle f'(t), g'(t) \rangle = \lim_{h \to 0} \frac{1}{h}(\mathbf{r}(t + h) - \mathbf{r}(t)) \]

we can interpret \( \mathbf{r}' \) in terms of a scaled version of a difference of two vectors. This is the easiest way to see that \( \mathbf{r}' \) will actually point in a direction tangent to the space curve of \( \mathbf{r}(t) \).

- As in the movie we saw in class last time, we can imagine that if we view \( \mathbf{r}(t) \) as the location of a moving particle, then the tangent vector \( \mathbf{r}'(t) \) points out of the front of that particle like a headlight, pointing in the direction that the particle is headed at that moment. In this context, we can also refer to the tangent vector \( \mathbf{r}'(t) \) as the velocity vector \( \mathbf{v}(t) = \mathbf{r}'(t) \) in analogy to the fact that \( s'(t) \) is the velocity function is \( s(t) \) is the position of an object moving along a line.
Also, as in the one variable case, differentiation allows us to find the tangent line to the space curve of a vector valued function. The tangent line to the space curve of \( \mathbf{r}(t) \) at \( P = \mathbf{r}(t_0) \) is defined to be the line through \( P \) parallel to the tangent vector \( \mathbf{r}'(t_0) \). (Often, one prefers to find the unit tangent vector \( \mathbf{T} \), which is simply found by dividing by the length.) So, the tangent line to \( \mathbf{r}(t) \) at \( t = a \) can be written in vector form as

\[
\mathbf{L}(t) = \mathbf{r}(a) + t\mathbf{r}'(a).
\]

What does it mean to say a line is “the tangent line” to a space curve at a point? This means that if you zoom in on the space curve of \( \mathbf{L} \) and \( \mathbf{r} \) at the point \( \mathbf{r}(a) \) they will start to look indistinguishable. This is because they not only both go through that point but because the curve and the line are also moving “in the same direction” at that point.

**Question 2:** Find the tangent line and unit tangent vector for \( \mathbf{r}(t) = (1 + t^3)i + te^{-t}j + \sin(2t)k \) at the point where \( t = 0 \).

The length of the velocity vector (also known as the tangent vector) is an important and physically simple thing as well. The length of the tangent vector (as you know, always a non-negative scalar) is the speed of the moving particle. So, if the particle is moving quickly at time \( t \), then \( v(t) = |\mathbf{v}(t)| \) is a big number and if the particle is practically stationary then it will be quite small.

**Main Idea:** The velocity is a vector that gives the direction as well as the speed. The speed is just the length of the velocity vector and it is always a non-negative scalar. You can get the speed from the velocity but not vice versa.

**Question 3:** Is \( \mathbf{r}(t) = \langle 1 + t^3, t^2 \rangle \) smooth on the interval \( I = (-1, 1) \)?

So, why do we need to assume that \( \mathbf{r}'(t) \) exists and is non-zero in order to say that the space curve of \( \mathbf{r}(t) \) is smooth? It is simply that at any sharp point on a parametrized curve, either \( \mathbf{r}'(t) \) does not exists or it is zero. (Consider \( \mathbf{r}(t) = \langle t^3 + 1, t^2 \rangle \) for an example.) So, if we rule out these things we can be sure that the curve is actually “smooth” in the usual English sense of the word.

**Homework**

**Section 13.2:** Read the section and do these problems: 9–14, 17–16

**Section 13.4:** Read page the bottom of 838 and the top of 839. Then try problems 3–6 and 9–12 without worrying about “acceleration”, just velocity and speed.