Key Ideas: Cross and Triple Products

- Today we’ll learn about the cross product of two vectors \( \mathbf{a} \) and \( \mathbf{b} \). In fact, we will learn three different definitions of this product. It requires a mathematical proof to say with certainty that all three of these definitions are equivalent. However, I will not discuss the details of this proof. (Read Section 12.4 carefully if you want to see it.)

- **Note:** Unlike the dot product, the cross product is only defined for vectors in three dimensions. (In other dimensions, it takes the form of the “wedge product”, which we’ll only learn about briefly on the last day of class.)

- **Determinental Definition:** The determinant is a technique from linear algebra which we introduce here for computational convenience. This is the technique that most people use to find the cross product. First, you need to know the determinant of order two:

\[
\begin{vmatrix}
  a_1 & a_2 \\
  b_1 & b_2 \\
\end{vmatrix} = a_1b_2 - a_2b_1
\]

and then we can find the order three determinant as

\[
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix} = a_1 \begin{vmatrix}
  b_2 & b_3 \\
  c_2 & c_3 \\
\end{vmatrix} - a_2 \begin{vmatrix}
  b_1 & b_3 \\
  c_1 & c_3 \\
\end{vmatrix} + a_3 \begin{vmatrix}
  b_1 & b_2 \\
  c_1 & c_2 \\
\end{vmatrix} .
\]

Now we can say that the cross product of \( \mathbf{a} \) and \( \mathbf{b} \) is

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix}
  i & j & k \\
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
\end{vmatrix} .
\]

**Question 1:** Use this technique to compute the product \( \langle 1, 0, 2 \rangle \times \langle 0, 2, 1 \rangle \).

- **Direct Expression:** It is not difficult to compute from the determinental definition above that \( \mathbf{a} \times \mathbf{b} \) is just

\[
\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle .
\]

**Question 2:** Verify that this gives you the same thing as you found in the previous question.

- **Geometric Characterization:** The cross product \( \mathbf{a} \times \mathbf{b} \) is the vector that is orthogonal to both \( \mathbf{a} \) and \( \mathbf{b} \), that has the same length as the area of the parallelogram determined by \( \mathbf{a} \) and \( \mathbf{b} \) (\(|\mathbf{a}| |\mathbf{b}| \sin \theta\)), and points in the direction determined by the right hand rule (direction of thumb when fingers pass first through \( \mathbf{a} \) and then \( \mathbf{b} \)).

**Question 3:** Use the Geometric Characterization to determine \( \mathbf{i} \times \mathbf{j} \) and \( \mathbf{i} \times \mathbf{k} \).
• **Corollary:** It is only from the last form of the definition that we can see that \( \mathbf{a} \times \mathbf{b} = \mathbf{0} \) if and only if the two vectors are parallel.

• Look at the boxed formulas on page 790 for properties of the cross product. Be careful! This is probably your first experience with a product which is not commutative or associative!

**Question 4:** Compute \( \mathbf{j} \times \mathbf{i} \) using any method you prefer and compare to your answers in Question 3. What can you conclude from this?

**Question 5:** Compute \((\mathbf{i} \times \mathbf{i}) \times \mathbf{j}\) and \(\mathbf{i} \times (\mathbf{i} \times \mathbf{j})\). What can you conclude from this?

• Finally, if we combine the dot and cross products, we get the scalar triple product \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \). There is an easier way to compute it than literally computing the dot and cross products separately. In fact, you can just compute it as the order three determinant:

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.
\]

• The geometric significance of this number is that its absolute value is the volume of the parallelepiped (squashed box) that the three vectors generate (see page 791). This will not be important to us except that it shows clearly that the scalar triple product of three vectors is zero if and only if they lie in the same plane. (If they didn’t, then the volume would not be zero!)

• (Of course, there is a significance to the cross and triple products beyond what we’ve seen here. We will encounter these again in an applied setting once we have developed some calculus techniques in space.)

## The Three Products and Their Relationship to Angles

We have now learned three different kinds of products involving vectors. Each of them is related to angles between two vectors, but in a different way. Suppose \( \mathbf{v} \) and \( \mathbf{w} \) are two non-zero vectors then

1. The vectors \( \mathbf{v} \) and \( \mathbf{w} \) are parallel if one is a scalar multiple of the other.

2. The sign of their dot product tells the relationship of the angle between them to a right angle: if \( \mathbf{v} \cdot \mathbf{w} \) is positive the angle is less than 90°; if it is negative the angle is more than 90°, and if it is zero then the angle is exactly a right angle.

3. If \( \mathbf{v} \) and \( \mathbf{w} \) are parallel then \( \mathbf{v} \times \mathbf{w} = \mathbf{0} \) is the vector with all entries equal to zero. Otherwise, the cross product \( \mathbf{v} \times \mathbf{w} \) is a non-zero vector that is orthogonal to both \( \mathbf{v} \) and \( \mathbf{w} \).

## Homework

**Section 12.4:** Read the section and do these problems: 1–7, 8⋆(draw guidelines in picture to make it clear), 11–13, 19, 20⋆  

**Homework Collection:** Remember that all of the “starred” problems assigned between the first day of and last Friday are going to be collected tomorrow. However, the problems assigned today are not due until next week.