Key Ideas: Coordinates and Distance in $\mathbb{R}^3$-space

- Recall the number line that we all have used since elementary school. Each number has a location on the line. In that sense, a single number (for example $2$ or $\pi$ or $-798, 322.221$) is a coordinate on the line. That is why we say the line is one-dimensional.

**Question 1:** What does the equation $x = 4$ mean on the line?

**Question 2:** What about $x^2 + 1 = 2$?

- In contrast, we need to know two numbers to be able to identify a point in the plane. Any two numbers correspond to a point, and any point corresponds to a unique pair of numbers. We often call those numbers $x$ and $y$, and these are the coordinates for the $(xy)$-plane.

**Question 3:** What is the graph of the equation $x^2 + y^2 = 9$ in the plane?

**Question 4:** What about the equation $x = 5$? (Note especially the significance of the fact that no $y$ appears here.)

**Question 5:** Then, what is the graph of $x^2+1 = 2$ in the plane? (Observe how one dimensional objects get stretched out straight into two dimensions when one of the variables is missing from the equation.)

- These two coordinates were enough for calculus in all of your previous calculus classes because all of the graphs you ever considered were curves in the plane, and all of the functions you considered just took a single number as input ($x$) and gave you a single number as output ($y$). However, functions that we have to deal with in the real world are rarely this simple. Either the input or the output may have to be in a higher dimensional space. In fact, if you continue in mathematics beyond this class, you will likely see 5, 10 or even 2002 dimensional spaces...but not in this class. We will now introduce three-dimensional space as the next generalization of the line and the plane, and this will be our primary working environment throughout the class.

- (3-dimensional) Space usually is described using the coordinates $x, y$ and $z$. By convention, the $z$ axis is vertical (joint between two walls) and the $xy$-plane is horizontal (it is the floor). Moreover, we choose to use the right-hand rule to determine which is the $x$ and which is the $y$ axis (see page 765).
There are three coordinate planes in $\mathbb{R}^3$: the $xy$-plane (where $z = 0$), the $xz$-plane (where $y = 0$) and the $yz$-plane (with $x = 0$).

**Question 6:** What is the graph of $x + z = 4$ in space?

**Question 7:** What does $x^2 + y^2 = 9$ look like in space?

**Question 8:** What are the points that satisfy $z = 4$ and $x = 1$?

- The distance $|P_1P_2|$ between the point $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by the formula

\[ |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \]

**Question 9:** What is the graph of $x^2 + y^2 + z^2 = 9$ in space?

**Question 10:** What about the inequality $x^2 + y^2 + z^2 \leq 9$?

- In general, the formula for a sphere with center at $(h, k, l)$ and radius $r$ is

\[(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.\]

Pay special attention to the next two remarks since they will help you with your homework:

- Later this week we are going to learn a new sort of algebra, one that works especially well with three dimensional space. Using this algebra, for instance, it is easy to prove that that if the sides of a triangle have lengths $a$, $b$ and $c$ then it is a right triangle if and only if and $a^2 + b^2 = c^2$. (Of course, you already know that the equation is satisfied if it is a right triangle, that’s the Pythagorean Theorem. However, you might not know that the converse is true as well...and it’s pretty easy to prove with the algebraic geometry of vectors.) I’ll prove this for you later, but you might want to use this fact in tonight’s homework. (That’s a hint, by the way.)

- Substituting a constant value into one of the variables in an equation and leaving the other two “free” is a trick we will find useful throughout this course. It is a way of reducing a three dimensional problem to a two dimensional one. For example:

**Question 11:** The points satisfying $z = x^2 + y^2 - 1$ form a surface in space. What geometric shapes are formed as the intersection of this surface with the plane $z = 0$? What about its intersections with the other two coordinate planes?

**Homework**

**Section 12.1:** Read the section and do these problems: 3, 5, 7, 8*, 11, 12*, 23–36
(The problems with stars will be collected next Tuesday.)