Math 221 Handout: April 8, 2009

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Key Ideas: Parametrized Surfaces

- Most of the surfaces we have considered so far have been graphs of functions. That is, given a function \( f(x, y) \) we look at the set of points satisfying \( z = f(x, y) \). A more useful way to think of this relationship for today's lecture is that we actually associate a point on the surface to each point \((x, y)\) in the domain of \( F \):

\[
(x, y) \mapsto (x, y, f(x, y)).
\]

In this description it seems more like a coincidence that the first two coordinates of the point on the surface happen to be the same as the point we started with. More generally, we can describe surfaces parametrically by associating a point on the surface to a point in some domain \( D \) in the plane in more general ways.

- **Parametrized Surfaces**: Let \( D \) be a domain in the \( uv \)-plane on which the continuous functions \( x(u, v) \), \( y(u, v) \) and \( z(u, v) \) are all defined. Then the parametric surface \( S \) defined by these functions is the set of points \((x(u, v), y(u, v), z(u, v))\) where \((u, v)\) run over all of \( D \). Alternatively, we could say it this way. Let \( r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k \) and then \( S \) is “traced out” by the tip of the position vector \( r \) as \((u, v)\) moves throughout the region \( D \).

(Note that the graph of \( z = f(x, y) \) is a special case of this for which \( x = u, y = v \) and \( z = f(u, v) \).)

- **Grid Lines**: We can easily imagine a grid on the \( uv \)-plane such that along each line either \( u \) or \( v \) is held constant. We would just look at the vertical and horizontal lines that are some fixed distance apart. It is often useful to look at the images of these lines on the surface \( S \). They help you to visualize \( S \) from a graph, and to make a graph from a formula.

**Question 1:** Find a parametric representation of the sphere \( x^2 + y^2 + z^2 = 1 \). What are the grid lines?

**Question 2:** Find a parametric representation for the cylinder \( x^2 + y^2 = 4 \) with \( 0 \leq z \leq 1 \).

- **Surfaces of Revolution**: You probably remember encountering surfaces of revolution in your previous classes. It turns out that these have a simple description as parametrized surfaces. In fact, for any function \( f(x) \) defined on an interval \( a \leq x \leq b \), the surface of revolution around the \( x \)-axis is given by

\[
 x = u, \quad y = f(u) \cos(\theta), \quad z = f(u) \sin(\theta).
\]

**Question 3:** Find parametric equations for the surface generated by revolving \( y = \sin x \) for \( 0 \leq x \leq 2\pi \) around the \( x \)-axis. What do the grid lines look like?
• **Differentiating r:** We can take partial derivatives of \( r \) with respect to either \( u \) or \( v \). We will write these, following a previously introduced notation, as \( r_u \) and \( r_v \). In the next two applications, we will actually be interested in the cross product \( r_u \times r_v \).

• **Smooth Parametrizations:** As you can imagine, there are many different ways to parametrize the same surface. We will be especially interested here in *smooth parametrizations*. A parametrization is said to be smooth if the vector \( r_u \times r_v \) is never the zero vector on the domain \( D \). The name comes from the fact that if the surface is not what you would like to call “smooth” in English – if it has sharp corners or edges – then it has no smooth parametrizations according to this definition.

• **Tangent Planes:** The vector that arises in the definition of “smooth” actually has a direct application in finding the tangent plane to a parametrized surface. Note that \( r_u \) and \( r_v \) are tangent vectors to the surface at each point. Consequently, their cross product \( n = r_u \times r_v \) is a normal vector to the tangent plane.

**Question 4:** Find the tangent plane to the surface with parametric equations \( x = u^2 \), \( y = v^2 \), \( z = u + 2v \) at the point \((1, 1, 3)\).

• **Surface Area:** The length of \( r_u \times r_v \) also has a significance. It is, as you may recall, the area of the parallelogram generated by these two vectors. It is this property that leads us to conclude that the surface area of \( S \) is computed as follows:

\[
A(S) = \iint_D |r_u \times r_v| \, dA.
\]

(NB: The “\(dA\)” here means \(du \, dv\) or \(dv \, du\) in the \(uv\)-plane.)

• **Flashback:** In particular, if \( z = f(x, y) \) is the graph of a function, then the formula above for its surface area just turns into the formula we already knew.

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**Homework**

**Section 16.6:** Read the section and do problems 1–4, 11, 12, 13–16, 29, 30, 31–41