Key Ideas: Surface Area

- Double integrals are useful for more than figuring out volumes associated to graphs. (If that was their only use, then they would have little application in science and engineering. However, as I’ve shown you in class, they do get used.) To understand this more general use, one should think again of the definition of double integrals in terms of Riemann sums; they involve evaluating the function at a point, multiplying by a little bit of area, and then adding up all of these products. It is almost like adding up all of the values of the function on the region, but that step of multiplying by the little area is key.

For example, you can read in section 15.5 about how the integral of the density function gives the total mass of an object and the integral of the probability distribution function gives the total probability. A practical example of this sort (for biologists) is the following: If \( f(x, y) \) gives the density of bacteria per unit of area, then the integral of \( f \) over \( R \) gives the total number of cells in the region \( R \). Suppose, for instance, the function \( f \) was constant (something like 125 bacterial cells per square inch). Then multiplying by the area would give the total number of cells, right? Similarly, if \( f \) is not constant, you could get a good estimate of the total number of cells by using that same procedure on little squares (chosen so small that the value of \( f \) does not vary much) and adding up all of these products. In fact, to estimate the number of bacterial cells in a petri dish, my wife who is a molecular biologist essentially does a Riemann sum where the region is broken up into little squares by placing the dish over a piece of graph paper and the “evaluation” of the function is achieved by literally counting cells with a microscope.

- The only application of this sort that we will really consider here is the surface area of part of the graph of \( z = f(x, y) \) in Section 15.6. Consider the graph of \( z = f(x, y) \) over the rectangular region \( R \) and imagine \( R \) broken down into lots of smaller rectangles as in the definition of the double integral. What would we add up to get the surface area? Since the Riemann sum is only an approximation anyway, if the function is differentiable on \( R \) it is good enough for us to add up the area of the piece of the tangent plane over each small rectangle, which should be close to the actual surface area and will only get more accurate as the rectangles get smaller.

- So, now we need a formula for the area of the piece of the tangent plane over the rectangle with a corner at \((x_i, y_j)\) and sides of length \( \Delta x \) and \( \Delta y \). Notice that this piece is the parallelogram with the tangent vectors to the graph in the \( x \) and \( y \) directions as two sides. So, in particular, one side of this parallelogram is

\[
a = \Delta x \mathbf{i} + f_x(x_i, y_j) \Delta x \mathbf{k}
\]

and another side is

\[
b = \Delta y \mathbf{j} + f_y(x_i, y_j) \Delta y \mathbf{k}.
\]
Then, what is the area of this parallelogram? You may recall that we can find it as $|\mathbf{a} \times \mathbf{b}|$, the length of the cross product! A simple computation then verifies that the area of this little piece is

$$\sqrt{f_x^2(x_i, y_j) + f_y^2(x_i, y_j) + 1 \Delta x \Delta y}.$$ 

Thus, the surface area of the whole piece of the graph over $R$ should be exactly

$$A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{f_x^2(x_i, y_j) + f_y^2(x_i, y_j) + 1 \Delta x \Delta y}.$$ 

- **Surface Area as a Double Integral:** Viewing this as a Riemann sum and allowing the consideration of more general domains $D$ we conclude that the area of the surface with equation $z = f(x, y)$ over $D$ (where $f_x$ and $f_y$ are continuous.) is

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA.$$ 

(Yes, if you want to, you can think of this as a special case of a “change of coordinates” with the integrand above being the Jacobian. We’ll see more about this later.)

**Question 1:** Find the surface area of the part of $z = x^2 + 2y$ above the triangle with vertices $(0, 0), (1, 0)$ and $(1, 1)$.

**Question 2:** Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

**Question 3:** How is the area of the piece of the graph of $z = ax + by + c$ above an arbitrary domain $D$ related to the area of $D$ itself?

### Homework

**Section 15.5:** Flip quickly through this section to get an idea of some applications of double integrals in physics and probability.

**Section 15.6:** Read the section and do problems 1–5, 6*, 7–12, 13a, 14a*.