From Last Time: Differential Equations

- Last time I did not quite get to finish discussing differential equations. The point is that a differential equation can be used to model a real situation. For instance, the equation \( u_{tt} = c^2 u_{xx} \) for a function \( u(x, t) \) of the variables \( x \) and \( t \) models the dynamics of a vibrating guitar string. Here we think of \( x \) as being a position along the string, \( t \) as being a time and \( u(x, t) \) as being the displacement of the point at \( x \) at time \( t \). When one looks at the solutions which are fixed at two endpoints (like a guitar string would be), the animations of the solutions do look like a string on a musical instrument. From this we can learn to understand how stringed instruments really work. (For instance, we can work out what note each string would play by its length and thickness, which would determine the constant \( c \). And we can understand the ‘harmonics’ that guitarists can play.) But, it is much more important to note that differential equations can help us discover things we do not already know about. In fact, this same equation was responsible for the discovery of invisible “radio waves” when J.C. Maxwell noticed it hidden in the formulas describing the interaction of electrons and magnets. It was because of the formula that we have microwave ovens, cell phones and lasers.

- In this class, you are really only responsible for being able to tell when a given function is a solution to a given differential equation. For instance, which of these is a solution to the differential equation in the preceding paragraph?

\[
\begin{align*}
    u(x, t) &= \sin(x + ct) - 2 \cos(x - ct) \\
    u(x, t) &= \frac{x}{ct} \\
    u(x, t) &= e^{cx} e^{ct} + 9xt.
\end{align*}
\]

Key Ideas: Tangent Planes and Linear Approximations

- An underlying theme of today’s lecture is the question: **What should “differentiable” mean for a function of two variables?** To address this question, let us remember what it meant in the one variable case.

- **Flashback:** In the case of a function \( y = f(x) \), we said that \( f \) was differentiable at \( x = a \) if \( f'(a) \) was defined. A consequence of this was that if the function was differentiable, we could zoom in on the graph at the point \((a, f(a))\) and what we saw would look almost indistinguishable from the tangent line \( y = f'(a)(x - a) + f(a) \). We could even take advantage of this similarity, using the simple linear formula above instead of the actual definition of \( f \) in order to approximate values of the function.

- **Tangent Planes:** By “zooming in” on a graph on the computer, we will see that natural generalization of the tangent line from the one-variable case is a tangent plane. In particular, if we pick a nice enough function \( z = f(x, y) \) and zoom in on the graph at the point \((a, b, f(a, b))\), we will see something that looks indistinguishable from a plane. If this is so, then we can easily say what plane it is:
So, given any function \( f(x, y) \), if \( f_x(a, b) \) and \( f_y(a, b) \) are defined then the above formula gives the equation of the tangent plane. Seems easy enough, right?

- **Uh oh!!:** However, there is a problem in paradise. Unlike the one-variable situation in which anytime the derivative existed you knew that the tangent line was there and was a good approximation for the curve, it is possible for both of the partial derivatives to exist even if the graph does not look like a plane near \((a, b)\)! To understand this better we need to learn just a bit more about limits of functions like \( f(x, y) \).

- As in Calculus I at this college, we will not give the ideas of “limits” and “continuity” their deserved coverage in this class. These are really important ideas. If calculus was a car, these would be the engine and the fuel. Since we will not be learning enough about them, it is as if we are learning to drive the car here but not how to build it from scratch. I will tell you just a little bit about these ideas (enough to understand how the car works or maybe even to fix a minor breakdown) but for more details you will have to read the section yourself or take a course in analysis.

- For a function \( f(x, y) \) of two variables and fixed real numbers \( x, y \) and \( L \), the statement
  \[
  \lim_{(x, y) \to (a, b)} f(x, y) = L
  \]

  means *loosely* that the output of the function \( f \) “approaches” the value \( L \) as the input values get closer to \((a, b)\).

  - More specifically, it means that you can ask me to give you a little circle centered at the point \((a, b)\) for which the first \( n \) digits of \( L \) and \( f(x, y) \) agree if I take \((x, y)\) to be any point in that circle (other than the center). If you ask me to get the first 100 digits to agree, perhaps I could say that a circle of radius \( 0.4 \) has the property that all of the output values look like \( L \) to the first 100 digits. If you ask for more digits (like 2 million) I would probably have to take a smaller circle. But, the claim is that I could get as many digits as you ask for as long as I make the circle small enough.

**Question 1:** Estimate \( \lim_{(x, y) \to (0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} \) and \( \lim_{(x, y) \to (0, 0)} \frac{3x^2y}{x^2 + y^2} \) by plugging in values of \((x, y)\) near \((0, 0)\).

**Question 2:** In contrast try values of \( \frac{xy}{x^2+y^2} \) near \((0, 0)\) but either on the line \( y = 0 \) or on the line \( y = x \).

- **A way to show the limit does NOT exist:** If the value of \( f \) approaches different values when \((x, y)\) approaches \((a, b)\) along different paths, then the limit does not exist.

- Then, “continuous” means that the above equation and \( f(a, b) = L \) are both true. (So, the value the function approaches is equal to the value it has at that point.)
• The function \( f(x, y) = \frac{xy}{x^2 + y^2} \) gives a good example of a function that does not look like its tangent plane if we just add the definition \( f(0, 0) = 0 \) so that it is defined everywhere. Note that \( f_x(0, 0) = 0 \) and \( f_y(0, 0) = 0 \), but it does not look like the tangent plane \( z = 0 \) if you are on the line \( y = x \!.

• **Definition:** We want to rule out all of the “bad places” on the graph of a function at which the graph does not look locally like a plane. Therefore, unlike the one-dimensional case, we say by **definition** that a function \( f(x, y) \) is **differentiable** at \((a, b)\) if it looks like the tangent plane at that point. (This is a loose way of saying the same thing as theorem 7 on page 911.)

• **How to tell if it is differentiable?:** This criterion (which is not equivalent to the definition, it is weaker) is easier to use than the real definition: If \( f_x \) and \( f_y \) are continuous at \((a, b)\) then \( f \) is differentiable at \((a, b)\).

• If a function is differentiable, then you can use the tangent plane as a **linear approximation** to the function itself.

**Question 3:** Show that \( f(x, y) = xe^{xy} \) is differentiable at \((1, 0)\) and use linear approximation to estimate \( f(1.1, -.1) \).

• Although we lose the visual aspect of the geometry, everything we’ve said continues to apply in the higher dimensional cases as well. For example, for a function of 3-variables we can say that \( f \) is differentiable at \((a, b, c)\) if

\[
  f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x-a) + f_y(a, b, c)(y-b) + f_z(a, b, c)(z-c).
\]

• I am not going to worry about the definition and notation of differentials that is covered in this section. There is a good reason for this notation in a class like differential topology or differential geometry, but here it is just another way to write the equation of the tangent plane.

• **Implicit Differentiation:** See Chapter 14.3 (p. 900) for an example of implicit differentiation in the multi-variable case. I will not discuss it much for the simple reason that we will soon “discover” a simpler technique for doing the same thing.

---

**Homework**

**Section 14.4:** Read the section carefully and do these problems: 1–3,4*,5–6, 11–17,18* (you do not need to draw a graph), 19.

**HW Due Tomorrow:** Remember that tomorrow I’ll be collecting the following homework: 14.1 30, 42 and 14.3 34, 46, 66.