Key Ideas: Dot Products and Geometry of \( \mathbb{R}^n \)

- If a vector \( \mathbf{z} \) is orthogonal to every vector in a subspace \( \mathbf{W} \) then we say it is orthogonal to \( \mathbf{W} \). (For example, \( \mathbf{e}_3 \) is orthogonal to \( \mathbf{W} = \text{span}\{\mathbf{e}_1, \mathbf{e}_2\} \) in \( \mathbb{R}^3 \).) The set of all vectors orthogonal to \( \mathbf{W} \) is called the orthogonal complement of \( \mathbf{W} \) and is written as \( \mathbf{W}^\perp \).

**Question 1:** Show that \( \mathbf{W}^\perp \) is a subspace. In particular, verify that it contains the zero vector, is closed under scalar multiplication and vector addition.

- In practice, it is not so easy to show that a vector is orthogonal to every element of \( \mathbf{W} \). To check if a vector \( \mathbf{v} \) is in \( \mathbf{W}^\perp \), you can just check if \( \mathbf{v} \) is orthogonal to every vector in a set that spans \( \mathbf{W} \).

- **Theorem 3** says

\[
(\text{row } \mathbf{A})^\perp = \text{Nul } \mathbf{A} \quad (\text{Col } \mathbf{A})^\perp = \text{Nul } (\mathbf{A}^\top).
\]

**Question 2:** Let \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \). If \( \mathbf{W}_1 = \text{span}\{\mathbf{v}_1\} \) and \( \mathbf{W}_2 = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \), use Theorem 3 to find bases for \( \mathbf{W}_1^\perp \) and \( \mathbf{W}_2^\perp \). Interpret your results geometrically.

- I cannot see where the book proves this fact, but it easily follows from Theorem 3 and our earlier statement about the sum of the dimensions of the column and null spaces of a matrix that:

\[
\text{If } \mathbf{W} \subset \mathbb{R}^n \text{ then } \dim \mathbf{W} + \dim \mathbf{W}^\perp = n.
\]

- If you look back at my notes from the first day of class, you’ll find that I said we would be using the Pythagorean Theorem in \( 928735 \)-dimensional space. We have not yet found the use for it, but we can now state and prove it. I will take you through it in a couple of steps below.

**Question 3:** Fill-in the blanks with numbers to make the equality true:

\[
||\mathbf{u} + \mathbf{v}||^2 = \_||\mathbf{u}||^2 + \_\mathbf{u} \cdot \mathbf{v} + \_||\mathbf{v}||^2.
\]

(Hint: Rewrite the length on the left side as a dot product, distribute the multiplication, and then turn some of the dot products back into lengths.)
• We can interpret the lengths above as being sides of a triangle as follows: Draw the vector \( \mathbf{u} \) with its tail at the origin, then draw the vector \( \mathbf{v} \) with its tail at the tip of the first vector. The vector \( \mathbf{u} + \mathbf{v} \) is then the vector which connects the open tail of \( \mathbf{u} \) to the open head of \( \mathbf{v} \). Together they form a triangle. According to our answer to the question above, the square of the length of one side is equal to the sum of the squares of the lengths of the other sides plus twice their dot product...sound vaguely familiar?

• The Pythagorean Theorem (Math 203 Version): Recall that we are replacing the notion of a “right angle” in \( \mathbb{R}^n \) by orthogonality. So, if \( \mathbf{u} \cdot \mathbf{v} = 0 \) the above theorem turns into a familiar statement about right triangles. The analogue of the usual Pythagorean theorem is:

\[
\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal if and only if } ||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2.
\]

---

**Homework**

**Section 6.1**: Read the section and then do these problems: 23, 24, 25–27, 28

**About Problem #24**: We want to show

\[
||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2.
\]

Look at what this means geometrically: Given a parallelogram, the sum of the squares of the lengths of the two diagonals is the sum of the squares of the lengths of the four sides. That’s kind of cool, and it is true for any parallelogram (unlike the Pythagorean theorem which is only true for right triangles). In this problem, we want you to prove this fact as we did the Pythagorean theorem in class. That is, rewrite the lengths in terms of dot products and show that the two sides of the equation really are equal.