Dot Products and Geometry of $\mathbb{R}^2$

- **Motivating Story:** You can read in the introduction to Chapter 6 about how the National Geodetic Survey ended up having to solve an equation of the form $Ax = b$ for a 1,800,000 by 900,000 matrix $A$. (The goal of their program is to determine the heights above sea level of a large number of landmark points in the US for surveying purposes.) Now, try to imagine their frustration if, after collecting all of the data to make this matrix, they found that it was an inconsistent system so that there is no such $x$! Actually, they pretty much expected it to be inconsistent, but they still sought to find a best approximate answer. In other words, they want to find an $x$ so that $Ax$ is as close to $b$ as possible. But, to talk about close you need an idea of distance in $\mathbb{R}^n$, which is a geometric concept. Moreover, to find this best solution you need to use the Pythagorean theorem and (in particular) right angles in $\mathbb{R}^n$, which are also geometric ideas. In this chapter, we will learn about the geometry of vector spaces (as given by linear algebra).

- Let's begin by talking about the geometry of $\mathbb{R}^2$, a vector space that you know very well, in terms of linear algebra. To be able to talk about geometry, you pretty much need to be able to measure three things: distance (or length), angles and area. If you have those, then you can talk about theorems of geometry like the fact that the sides of a right triangle satisfy $a^2 + b^2 = c^2$.

- **The Dot Product of Two Vectors:** Let $v$ and $w$ be vectors in $\mathbb{R}^2$. We define

$$v \cdot w = v^\top w.$$  

(In other words, it is the usual matrix product of the row vector $v^\top$ and the column vector $w$.) This is a scalar, so we usually write it without brackets. For example, if  

$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad w = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad v \cdot w = 1.$$  

- **The Length of a Vector:** For $v$ in $\mathbb{R}^2$ it is easy to check from the Pythagorean theorem that the length of the vector $v$ is  

$$||v|| = \sqrt{v \cdot v}.$$  

Let us consider this the definition of “length” for a vector. A unit vector is a vector of length one. You can make such a vector out of a given nonzero vector by scalar multiplication so that it still points in the same direction. Specifically, $u/||u||$ is always a unit vector pointing in the same direction as the nonzero vector $u$.

- This same idea of length then is used to determine the distance between two vectors:

$$\text{dist}(v, w) = ||v - w||.$$
• **Dot Products and Right Angles:** It follows from some basic trigonometry (see page 381) that the dot product of two non-zero vectors depends on the angle between them. In particular, the dot product in $\mathbb{R}^2$ is zero if and only if the angle between them is a right angle.

**Dot Products and Geometry of $\mathbb{R}^n$**

- For two vectors in $\mathbb{R}^n$ we define the *dot product* or *inner product* as

  $$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad u \cdot v = u^\top v = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

- The dot product (also sometimes called the inner product or the scalar product) has the following nice properties for any vectors $u, v$ and $w$ in $\mathbb{R}^n$:

  $$u \cdot v = v \cdot u, \quad (u + v) \cdot w = u \cdot w + v \cdot w,$$

  $$(cu) \cdot v = c(u \cdot v) = u \cdot (cv), \quad u \cdot u \geq 0, \quad u \cdot u = 0 \Rightarrow u = 0.$$  

- As it says above, $u \cdot u$ is always a positive number except when $u$ is the zero vector in which case it is zero. So, it makes sense for us to *define* the length of the vector $u$ to be the square-root of this number:

  $$||u|| = \sqrt{u \cdot u}.$$  

  A vector $v$ such that $||v|| = 1$ is called a unit vector. The vector $v / ||v||$ is a unit vector pointing in the same direction as $v$ as long as $v \neq 0$.

- To define a geometry on a space, a mathematician gives a procedure for finding the distance between any two points. The distance between the vectors $u$ and $v$ in $\mathbb{R}^n$ is

  $$\text{dist}(u, v) = ||u - v||.$$  

- We say that $u$ and $v$ are *orthogonal* to each other if $u \cdot v = 0$. This extends the notion of “right angle” to higher dimensional spaces, but is not quite the same as saying “form a right angle”. For example, note that according to this definition, $0$ is orthogonal to every other vector. On Friday, we will see that the Pythagorean Theorem is true in $\mathbb{R}^n$ if we use orthogonality to identify right triangles.

  ➤ **Warning:** We will only consider one definition of dot product. However, in general a mathematician is free to give the dot product a different definition, which then changes the geometry of the vector space. For example, in Minkowski space (for special relativity) some of the terms are subtracted rather than added, and in my paper “Orthogonal Polynomials and the Finite Toda Lattice” (*Journal of Mathematical Physics*, 1997) the dot product of two polynomials is in terms of the *integral* of their product.

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**Homework**

**Section 6.1:** We have not finished covering this section yet, but you should already be able to do these problems for homework on Friday: 1–5, 6, 7–17, 18