Key Ideas: Eigenspaces and Their Dimensions

1. If \( \lambda = c \) is an eigenvalue for the \( n \times n \) matrix \( A \), then the set of all vectors \( \mathbf{x} \) satisfying \( A\mathbf{x} = c\mathbf{x} \) is actually a subspace of \( \mathbb{R}^n \) which we call “the eigenspace corresponding to eigenvalue \( c \)”. (That this is true should be obvious from the fact that it is also \( \text{Nul} \left( A - cI_n \right) \) and we know that the null space is a subspace.)

2. The dimension of the eigenspace (which is the same as the number of non-pivot columns in \( A - cI_n \)) can be anything from 1 up to the multiplicity of \( c \) as an eigenvalue. (Remember: the multiplicity is the same as the highest power of \( \lambda - c \) in the characteristic polynomial \( p(\lambda) \).)

3. We can find a basis for the eigenspace in the same way as we find a basis for the null space of \( A - cI_n \).

4. Main Point: \( A \) is diagonalizable if and only if the dimensions of the eigenspaces for all of its distinct eigenvalues add up to \( n \). Taking a basis for each of the eigenspaces and putting them together as the columns of a matrix will give you an invertible \( n \times n \) matrix \( P \) such that \( A = PDP^{-1} \) for a diagonal matrix \( D \).

Key Ideas: Representations and Diagonalization

- Recall that if you have a vector space \( V \) with basis \( \mathcal{B} = \{ \mathbf{b}_1, \ldots, \mathbf{b}_n \} \) then you can talk about the coordinate vector \( [\mathbf{v}]_\mathcal{B} \) for any vector in \( V \). This allows us to turn things that do not initially look like vectors into column vectors of numbers. The point of today’s lecture is that we can similarly turn the linear transformations that act on them into regular matrices.

- The matrix representation of a linear transformation: Suppose that \( T \) is a linear transformation from vector space \( V \) with basis \( \mathcal{B} \) to vector space \( W \) with basis \( \mathcal{C} \). We define the matrix for \( T \) relative to the bases \( \mathcal{B} \) and \( \mathcal{C} \) to be the matrix

\[
M = \begin{bmatrix} [T(\mathbf{b}_1)]_\mathcal{C} & [T(\mathbf{b}_2)]_\mathcal{C} & \cdots & [T(\mathbf{b}_n)]_\mathcal{C} \end{bmatrix}.
\]

- The point is that the coordinate vector of the transformation applied to \( \mathbf{x} \) is the same as the result of multiplying the coordinate vector of \( \mathbf{x} \) by this matrix:

\[
[T(\mathbf{x})]_\mathcal{C} = M[\mathbf{x}]_\mathcal{B}.
\]

- This should look familiar, we previously did this when we made a matrix that did the same thing as a linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) (using the basis \( \{ \mathbf{e}_1, \ldots, \mathbf{e}_n \} \)). The only difference now is that we’re considering any vector space and any basis.
Now any linear transformation can be represented by a matrix, even if it does not sound “matrix-like” to start with. Remember Group Project #6?

**Question 1:** Differentiation is a linear transformation from $\mathbb{P}_2$ to $\mathbb{P}_1$. Using the standard bases for these spaces, what is the matrix representation $M$? What is the solution set of the equation $Mx = b$ with $b^T = [1 \ 1]$? What is the solution set of the equation $p'(t) = t + 1$?

**Question 2:** Let $L : \mathbb{P}_2 \rightarrow \mathbb{R}^3$ be the transformation

\[
L(p(t)) = \begin{bmatrix} p'(0) \\ p(1) \\ -2p(2) \end{bmatrix}.
\]

Find $L(t^2 - 2t)$ just by using the formula above. Now, find the matrix representation with respect to the standard bases for $\mathbb{P}_2$ and $\mathbb{R}^3$. Use matrix multiplication to find $L(3t^2 - 20t + 40)$.

- Very often, we will be interested in linear transformations from $V$ to itself, in which case one may want to use the same basis for $B$ and $C$. (In this case when there is only one basis to talk about, your book calls the matrix representation the $B$-matrix of $T$.)

**Question 3:** “Translation” (sliding the graph left or right) is a linear transformation on $\mathbb{P}_2$. We define it by saying $T(p(t)) = p(t + 1)$ (this slides the graph left one unit). What is $T(t^2)$? Find the $B$-matrix of $T$ where $B = \{1, t, t^2\}$ is the usual basis. What are the eigenvalues of this matrix, and what can you conclude? (Answer: There are some polynomials in $\mathbb{P}_2$ that stay the same when you apply $T$, but none that get multiplied by $\lambda$ for any $\lambda \neq 1$.)

- As I hinted last time, this gives us a way to understand diagonalization. Theorem 8 says if $A = PDP^{-1}$ with diagonal matrix $D$, then $D$ is the $B$-matrix of the transformation $x \mapsto Ax$ where $B$ is the basis of eigenvectors of $A$.

In other words, the difference between $A$ and $D$ is just the choice of basis: If we take the matrix $A$ (written in terms of the standard basis for $\mathbb{R}^n$) and rewrite it relative to a basis made up of eigenvectors for $A$, it becomes a diagonal matrix! (Think about why this makes sense.)

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**Homework**

**Section 5.4:** Read the section and then do these problems: 1, 2, 5, 6*, 7, 9–13, 14*, 15, 16, 23, 25