Key Ideas: Characteristic Equation and Diagonalization

- Let $M$ be an $n \times n$ matrix. Recall from last time that if we want to know whether a particular number $\lambda$ is an eigenvalue of $M$, what we can do is check if $\det(M - \lambda I_n)$ is zero. ($M - \lambda I_n$ is the matrix that looks just like $M$ except that it has $\lambda$ subtracted from every diagonal entry.) If the determinant is zero then $\lambda$ is an eigenvalue.

- What if we want to find the eigenvalues for $M$? The obvious answer is that we should think of $\lambda$ as a variable and solve the Characteristic Equation:

$$\det(M - \lambda I) = 0.$$  

- What sort of equation is that? The left-hand side is a polynomial of degree $n$ in $\lambda$. So let me mention a few important facts about solving the equation $p(\lambda) = 0$ for a polynomial $p$ of degree $n$.
  - If $p$ factors as
    $$p(\lambda) = (\lambda - r_1)^{M_1}(\lambda - r_2)^{M_2} \cdots (\lambda - r_m)^{M_m}p_2(\lambda)$$
    (here $p_2$ is either constant or is a polynomial of degree higher than one which cannot be factored) then we say the $r_i$’s are the roots of $p$ and the $M_i$’s are their multiplicities.
  - The sum of the multiplicities cannot be bigger than $n$.
  - You could factor it completely so that the sum of the multiplicities is exactly $n$ if you are willing to talk about complex numbers. That is, every polynomial $p(x) = x^n + $ lower terms factors as $p(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$ where $r_j$ are complex numbers of the form $r_j = x_ji + y_j$ with $i^2 = -1$.
  - If $n = 2$ you can use the quadratic formula to solve for the roots. If $n < 5$ there are more complicated formulas that always work. If $n \geq 5$ there is no standard procedure, so you have to be clever to find the roots and may not be able to do it at all.

**Question 1:** Find the eigenvalues of the matrix $A = \begin{bmatrix} -5 & 2 \\ -12 & 5 \end{bmatrix}$. What are the corresponding eigenvectors?

- **Similar Matrices:** We say that $M$ and $B$ are similar if $M = ABA^{-1}$. Using the properties of determinants you can easily show that if $M = ABA^{-1}$ then $\det M = \det B$. (You essentially saw this in your last test...remember?) This has many important consequences. Since $M - \lambda I$ and $B - \lambda I$ are also similar, $M$ and $B$ have the same characteristic polynomial and so they have exactly the same eigenvalues (and multiplicities) if they are similar (Theorem 4)!

- Diagonal matrices ($n \times n$ matrices that have entries $D = (m_{ij})$ but $m_{ij} = 0$ if $i \neq j$) are nice. Some of their nice properties are:
  - They commute with each other.
You can easily say what they do as a transformation since they just multiply each coordinate in a vector by a fixed constant.

You can easily compute their powers since $D^n$ is just the same thing as raising each diagonal entry to the $n^{th}$ power.

In other words, in many ways they are more like scalars than like matrices.

**Question 2:** Let $D_1 = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}$, $D_2 = \begin{bmatrix} \sqrt{33} & 0 \\ 0 & -e^3 \end{bmatrix}$ and $v = \begin{bmatrix} \sqrt{33} \\ e^{-3} \end{bmatrix}$. Find $D_1^4$, $D_2v$ and $D_1D_2 - D_2D_1$.

- **A Big Question:** Which matrices are similar to diagonal matrices?
- **Key Point of Section 5.3:** Most matrices are similar to diagonal matrices, and we can use this fact to our advantage. We say that such a matrix is **diagonalizable**, by which we mean it can be made diagonal using just a similarity transformation.

- **Key Point of Section 5.3 in Other Words:** If an $n \times n$ matrix has enough eigenvectors to form a basis for $\mathbb{R}^n$ then you can write the matrix in those coordinates and it will be diagonal!

- **How to Do It:** Find the eigenvalues and eigenvectors of $A$. If there are $n$ linearly independent eigenvectors then make a matrix $P$ with them as columns. The diagonal matrix with the corresponding eigenvalues on the diagonal (same order as in $P$) satisfies $A = PD P^{-1}$! (See examples 3 and 4 for messy details.)

- **An application (easy powers):** If $A = PD P^{-1}$ then $A^j = PD^j P^{-1}$.

**Question 3:** Diagonalize the matrix $A = \begin{bmatrix} -5 & 2 \\ -12 & 5 \end{bmatrix}$ and compute $A^{10}$.

- **Theorem 6:** If an $n \times n$ matrix has $n$ distinct eigenvalues, then it is diagonalizable.

  **Note:** If the eigenvalues are not distinct, the $n \times n$ matrix $A$ may or may not be diagonalizable depending on whether there are $n$ linearly independent eigenvectors. Even if it is not diagonalizable, there is something you can do that is almost as good, but we won’t be learning about it in this class. Here is an example of a matrix that is not diagonalizable:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$