Key Ideas: Eigenvectors and Eigenvalues

Consider the matrix and vectors
\[ A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \quad u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \]

Note that \( Au \) is a very different vector than \( u \) – it points in a different direction. On the other hand, \( Av = 2v \). (The matrix \( A \) just multiplied \( v \) by 2 like a scalar!) This is the special type of situation we wish to consider today.

**Definition:** If \( Av = \lambda v \) for a matrix \( A \), vector \( v \neq 0 \) and scalar \( \lambda \) then we say: \( v \) is an eigenvector of \( A \) and \( \lambda \) is an eigenvalue of \( A \).

Quick Observation: An \( m \times n \) matrix \( A \) has eigenvectors only if \( m = n \) (that is, if \( A \) is square)...why?

**Why do we care about eigenvectors?** As with most of the things in this class, you unfortunately will not see the real significance of this stuff unless you continue on to higher level classes in math, science or engineering. Just to give you some ideas: eigenvalues are of fundamental importance in the theory of waves and quantum mechanics...the different frequencies of a guitar string are eigenvalues and analogously the electron orbitals in an atom are the eigenvectors and the energies are the eigenvalues. Moreover, if you are modeling the dynamics of an animal population in the wild, the eigenvectors are the stable situations...the ones that won’t lead to extinction or over-population. In computer science, it is a tremendous time(=money) saver to first change to working in a basis containing the eigenvectors before performing a calculation. Did you know that armies stop marching in lock step before walking across a bridge? That’s because if they happen to all be walking at a speed which is an eigenvalue of a certain transformation the bridge will collapse! Remember the importance of \( e \) in calculus? Notice that \( e^{\lambda x} \) is an eigenfunction for the transformation \( D = \frac{d}{dx} \) with eigenvalue \( \lambda \).

A special case we already know: if \( v \) is a non-zero vector in \( \text{Nul} \ A \) then it is an eigenvector with eigenvalue zero! Hence we see that a square matrix \( A \) is invertible precisely when \( 0 \) is not an eigenvalue.

In fact, it is useful to expand the previous statement to handle the general case of eigenvalues. Note that saying \( \lambda \) is an eigenvalue of \( A \) is equivalent to saying that \( A - \lambda I_n \) has non-zero vectors in its null space.

**How to check if a given vector is an eigenvector:** Just multiply and see if it is! (That is, check to see whether \( Av \) is just a scalar multiple of \( v \). You can tell what scalar you would need to multiply by for the first component, and then just determine whether the same scalar would work for all of them.)
• **How to check if a given value \( \lambda \) is an eigenvalue:** By the remark above, we can just check if \( A - \lambda I_n \) has any non-zero vectors in its null space. In particular, if \( \det(A - \lambda I_n) = 0 \) (or equivalently, if the rank of \( A - \lambda I_n \) is less than \( n \)) then \( \lambda \) is an eigenvalue of \( A \).

• **How to find the eigenspace corresponding to a given eigenvalue \( \lambda \):** This is the same as finding \( \text{Nul}(A - \lambda I_n) \). That means (a) if you know what \( \lambda \) is you can make the matrix \( M = A - \lambda I \) and solve for the null space using row reduction and (b) since \( \text{Nul} \ M \) is a subspace, we know that any linear combination of eigenvectors for a given matrix \( A \) and eigenvalue \( \lambda \) is also an eigenvector for the same \( A \) and \( \lambda \). (Well, unless the linear combination results in the zero vector, since we’ve chosen not to call that an eigenvector.)

• **How to find the eigenvalues:** This can be very hard – as hard as solving polynomials of high degree – as we will see in the next section. All we can say so far is **Theorem 1:** If \( A \) is triangular then the diagonal elements are the eigenvalues.

**Question 1:** Let \( A \) be the matrix

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
0 & 0 & 6
\end{bmatrix}.
\]

By Theorem 1 we know that there are vectors \( v_1, v_2 \) and \( v_3 \) with the property that multiplying them by \( A \) is just the same as multiplying them by 1, 4 or 6 respectively. Does this mean that \( A \) acts like a scalar multiple on every vector \( v \)?

Well, it is true that \( v_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \) stays the same, \( v_2 = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}^T \) gets multiplied by 4 and \( v_3 = \begin{bmatrix} 16 & 25 & 10 \end{bmatrix}^T \) gets multiplied by 6 (\( Av_3 = \begin{bmatrix} 96 & 150 & 60 \end{bmatrix}^T \)). But look what happens to \( v_1 + v_2 = \begin{bmatrix} 3 & 3 & 0 \end{bmatrix}^T \) when you multiply it by \( A \). It becomes \( v_1 + 4v_3 = \begin{bmatrix} 9 & 12 & 0 \end{bmatrix}^T \) which is *not* a scalar multiple of the vector we started with. So, no...a matrix having eigenvalues does *not* mean that the matrix does nothing but scalar multiplication for *all* vectors. It means that for certain *special* vectors the action of \( A \) looks like scalar multiplication.

• In the previous example, the three eigenvectors are linearly independent. (So, we could use them as a basis for \( \mathbb{R}^3 \)...what would that do for us?) Is this a coincidence or part of a general fact?

• **Theorem 2:** If \( v_1, \ldots, v_r \) are eigenvectors corresponding to *different* eigenvalues, then they are linearly independent.

## Homework

**Section 5.1:** Read the section and then do these problems. 1–7,8\*, 9–11, 12\*, 13–20