Key Ideas: Rank and Row

- So far in this class, we have been writing vectors as columns. So an $m \times n$ matrix is made up of $n$ different column vectors each containing $m$ entries. However, there is no reason we can't do it the other way! So, let's think of an $m \times n$ matrix as a list of $m$ different row vectors each having $n$ elements! Then we define Row $A$ to be the space spanned by the row vectors of $A$.

- Here is an example:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$  

We know that the first three columns form a basis for Col $A$ because they are the pivot columns. So dim Col $A = 3$. What about dim Row $A$? It is also 3 because the non-zero rows are linearly independent.

- As surprising as it may seem, this is not a coincidence. For any matrix $A$ it is true that dim Col $A = \text{dim Row } A$! (The proof is elementary in reduced echelon form as above.)

- We define rank $A = \text{dim Col } A = \text{dim Row } A$ to be this number.

**Question 1:** Let $u$ and $v$ be vectors in $\mathbb{R}^n$. As we saw when we first learned how to multiply matrices, $uv^\top$ is an $n \times n$ matrix. Try it with $u = [1 \ 2]^\top$ and $v = [5 \ 7]^\top$. What is the rank of the matrix you get? Explain why it must be true for any $u$ and $v$ that the rank of the matrix $uv^\top$ is at most 1.

- Based on what we saw last time, we have that for an $m \times n$ matrix

$$\text{rank } A + \text{dim Nul } A = n.$$  

- This formula is useful for figuring out a bound on the size of Nul $A$ or Col $A$ (see homework problem 15) or for determining one exactly if the other is known (see problem 9).

**Question 2:** If $u_1$, $u_2$, $v_1$, $v_2$ are all in $\mathbb{R}^n$ then the matrix $B = u_1v_1^\top + u_2v_2^\top$ has rank at most 2. What can you say about the dimension of Nul $B$?

- It might be useful now to go back to linear systems now to think about how all of this applies. Suppose you have a system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{142}x_{42} = b_1$$

$$\vdots$$

$$a_{401}x_1 + a_{402}x_2 + \cdots + a_{4042}x_{42} = b_{40}$$

then you know that the homogeneous equation ($b_i = 0$) must have a solution set that is at least 2-dimensional. Moreover, if you know that it is exactly 2-dimensional, you know that the system is not inconsistent for any choice of $b$ because the column space is all of $\mathbb{R}^{40}_1$.

**Question 3:** Suppose a $4 \times 7$ matrix $A$ has four pivot columns. Is Col $A = \mathbb{R}^4$? Is Nul $A = \mathbb{R}^3$? Explain. (See #7, p 269.)

- Looking at $n \times n$ matrices, we then see a connection between the rank and invertibility. It is equivalent to say that rank $A = n$, that dim Nul $A = 0$ or that $A$ is invertible. (Theorem on page 262.)
• **Full Rank?** Let $A$ be an $m \times n$ matrix. An important question about $A$ is whether it has the biggest possible rank that it can have. The biggest rank $A$ can be is the *smaller* number of $m$ and $n$. If it does have this biggest possible rank, we say “$A$ has full rank” and otherwise “$A$ does not have full rank”. For a square matrix, having full rank is the same as being invertible. But, we now have a word that means something similar for rectangular matrices. If the columns of $A$ or the rows of $A$ are linearly independent, then $A$ has full rank and otherwise it doesn’t!

Side Note: Rank has become a very important theme in my mathematical physics research in the last few years. As I may have mentioned, I study the relationship between particles and waves. Back in 1999 I published a paper about quantum physics which used the fact that a matrix associated to the particles had rank one to prove “the Bethe Ansatz”. Since then, I’ve found that such “rank one conditions” are an unrecognized but key point in the connection between particles and waves. Many of my recent papers involve “rank”, and one even has “rank one conditions for rectangular matrices” in the title.

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**Homework**

**Section 4.6:** Read the section and then do these problems. $1–4^*$, $8–16^*$, 24, 25