Basis

- Recall that a set of vectors is called linearly independent if the only linear combination of them which is zero is the trivial combination. (Equivalently, they are linearly dependent if one of them is a linear combination of the others.) This definition works in any vector space. For example, \{1, t, t^2\} is a linearly independent set of vectors from \(P_2\). So is \{1, 1-t, 1-t^2\}. But \{1 + t^2, 1 - t, t + t^2\} is not linearly independent. Why?

(Answer: If \(p(t) = c_1 \times (1) + c_2 \times (t) + c_3 \times (t^2)\) (a linear combination of 1, t and \(t^2\)) then the only way \(p(t)\) could be the zero polynomial is if all of the \(c_i\) are zero. Similarly for 1, 1-t and 1-t^2. But \((1 + t^2) - (1 - t) - (t^2) \equiv 0\), so these three “vectors” are not linearly independent.)

- Definition: We say that the set of vectors \(B = \{b_1, \ldots, b_n\}\) is a basis for the vector space \(V\) if (a) \(B\) is linearly independent and (b) \(V = \text{span}\{b_1, \ldots, b_n\}\).

- Smallest Possible Spanning Set: One view of a basis is as the smallest set that spans a given space. For example, suppose you know that \(H = \text{span}\{v_1, \ldots, v_n\}\). This might not be a basis because some of the \(v_i\)'s might be combinations of the others. So, you can keep dropping the ones that are, and then when you reach a linearly independent set it will be a basis.

- Largest Possible Independent Set: Another way to view a basis is as the biggest set of vectors that is linearly independent. If you have a set of vectors from a space \(H\) which is linearly independent but adding any other vector from \(H\) makes it dependent, then it is a basis for \(H\).

- No set of less than \(n\) vectors can be a basis for \(\mathbb{R}^n\) because they don’t span it. No set of more than \(n\) vectors can be a basis for \(\mathbb{R}^n\) because they cannot be linearly independent.

- If you have \(n\) vectors from \(\mathbb{R}^n\) then they are a basis as long as they are linearly independent. For instance, the standard basis \(e_1, \ldots, e_n\) (the columns of the identity matrix) are a basis. But so are the vectors in example 5 on page 238. In fact, every vector space has many different possible bases.

- Remember, a subspace is itself a vector space. (It is just a vector space that is contained in some other vector space.) That means we can also ask for a basis for vector spaces like \(\text{Col} A\):

\[ \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 0 & 2 & 4 \\ 3 & -1 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

- Theorem 6: The pivot columns of \(A\) form a basis for \(\text{Col} A\). (Also, when we write the general solution to \(Ax = 0\) in parametric vector forms, the vectors that appear in the expression – those that are multiplied by the free variables – are a basis for \(\text{Nul} A\).)

Question 1: Use the fact that \(M = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 0 & 2 & 4 \\ 3 & -1 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\) to find bases for \(\text{Col} M\) and \(\text{Nul} M\).

- All of these same ideas work in other vector spaces as well. For example \(P_2 = \text{span}\{1, t, t^2\}\) and since these are linearly independent, they form a basis for the vector space of polynomials of degree at most 2.

Bases and Coordinates

- What is the significance of bases for vector spaces? I hope this will help you better understand why \(P_n\) is also a vector space. In fact, as we will see, from a certain point of view \(P_n\) is one way of interpreting \(\mathbb{R}^{n+1}\)!
Suppose you have a basis $\mathcal{B}$ for a vector space $V$ and it has $n$ vectors in it. Then we can coordinatize $V$, representing each element of $V$ as an element of $\mathbb{R}^n$. Specifically, since every element of $V$ can be written as a linear combination of the elements of $\mathcal{B}$ we can just say “how much” of each of the basis elements we need to get any desired $x$ in $V$ and call these numbers its “coordinates”. Remarkably, you can then work with these vectors in place of any other representation of $V$ and do all of the things we generally do with vectors (solve by row reduction, take determinants, find matrix representations and inverse matrices, look for pivot columns, etc.)

- This is a consequence of Theorem 7 which says that given a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$, then any vector $x$ in $V$ can be written as
  $$x = c_1b_1 + \cdots + c_nb_n$$
  for a unique set of constants $c_i$. Then we define the coordinate vector of $x$ (relative to $\mathcal{B}$) by

  $$[x]_\mathcal{B} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$  

- This gives us a way to assign vectors as coordinates for abstract spaces like $\mathbb{P}_n$. For instance, if $B = \{1, t, t^2\}$ is the standard basis of $\mathbb{P}_2$ then we can describe polynomials in terms of their coordinate vectors:
  $$[1-t^2]_B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad [c + bt + at^2]_B = \begin{bmatrix} c \\ b \\ a \end{bmatrix}.$$  

- **A Use For Coordinate Vectors:** If I give you a set of vectors, how would you check if they were linearly independent? One way would be to check whether every column is a pivot in the matrix that they form. What if I gave you a set of polynomials and asked you whether they were independent? What you can do is translate them into coordinate vectors and then just check whether those vectors are linearly independent by row reduction! (This is a consequence of the fact that the coordinate map is a linear transformation, and so all of the vector calculations in one space are accurately reproduced in the other. See page 251-252 for discussion and examples.)

- This also gives us a way to choose different coordinates for a vector space. Even in $\mathbb{R}^n$ (where vectors already look like columns of numbers) we sometimes want to consider writing coordinates in terms of another basis. This is often the easiest way to compute something. If some computation seems difficult, consider changing the coordinates to one in which it is easier!
**How to change coordinates in** $\mathbb{R}^n$: Multiplication by $P_B = [b_1 \ldots b_n]$ converts coordinates relative to $B$ to standard coordinate in $\mathbb{R}^n$ and its inverse converts the other way. (See page 249.)

Here is an example. Note that $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ form a basis $B$ for $\mathbb{R}^2$. The coordinate vector for the vector $x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ in terms of this basis is

$$[x]_B = P_B^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

because $1 b_1 + 1 b_2 = x$. Conversely, if we want to know what point $x$ in $\mathbb{R}^2$ has coordinate vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in terms of the basis $B$ it would be

$$x = P_B \begin{bmatrix} 1 \\ -1 \end{bmatrix} = b_1 - b_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$  

**An Example where an unusual choice of coordinates is nicer:** The polynomials 
$p_1(t) = 2 + t + 3t^2$ and $p_2(t) = 3 + t + 4t^2$ span a subspace of $\mathbb{P}_2$, $V = \text{span}\{p_1(t), p_2(t)\}$. Question: Is the polynomial $q(t) = 2 + 4t + 3t^2$ in $V$? We can apply the ideas learned today to check this using methods from earlier in the course. Specifically, we can coordinatize by choosing a basis for $\mathbb{P}_2$...but which basis should we pick? Your first thought would probably be to use the basis $B = \{1, t, t^2\}$ in which case we want to know whether the vector $[q(t)]_B$ is in the space spanned by $[p_1(t)]_B$ and $[p_2(t)]_B$ where you can easily see that

$$[p_1(t)]_B = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad [p_2(t)]_B = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \quad [q(t)]_B = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}.$$  

Well, I still can’t tell just by looking what the answer to the question is...but we could check by making a matrix and row reducing. However, if we happen to pick a different choice of basis for $\mathbb{P}_2$, the question might be obvious. For example, suppose we work in the basis $C = \{1 + t + 2t^2, 1 + t^2, t\}$ for $\mathbb{P}_2$, then we see that

$$[p_1(t)]_C = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad [p_2(t)]_C = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad [q(t)]_C = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$  

Now it is clear that the third vector can not be written as a linear combination of the first two, and so the same is true of the polynomials. (After all, these are not really different objects, just coordinates for the polynomials we started with.)

**Homework**

**Section 4.3:** Read the section and do these problems: 1-15, 16⋆, 33, 34  
**Section 4.4:** Read the section and then do these problems: 1–13, 14⋆, 27–32