Review: What is a subspace?

- If you do not get this concept it may be difficult for you to understand much of what we will learn later in the class. Today I will attempt to explain it one more time. After this, if you do not understand this concept, I ask you to please visit me in my office so that we can discuss it one-on-one.

- Forgetting about what vectors look like, forgetting about how we write them, the important thing is how they behave. In particular, there are just three things that make vectors vectors:
  1. A special one of them (called “the zero vector”) can be added to any other vector without changing it.
  2. If you have any two vectors (in the same space) and add them together, you get another vector in the same space.
  3. If you have any vector and you multiply it by any given real number you get another vector in the same space. We call any set a vector space if it has these three properties. For example, the set $\mathbb{P}_n$ of all polynomials of degree $n$ or less is a vector space. (In $\mathbb{P}_n$, we talk about “the zero polynomial”...the one that can be added to another polynomial without changing it...the one that has the value zero no matter what number you plug into it...the one with all coefficients equal to zero.)

- Then, we can ask if a subset of a vector space is a subspace. That is, if I choose some (but not necessarily all) of the elements from a given vector space, do we get a new vector space? The answer depends on whether the three properties above apply to the subset as well.

- For instance, we can consider the subset of $\mathbb{P}_5$ made up of all of the polynomials whose graphs go through the origin. This is a subspace because the zero polynomial is one of these, and because scaling a function preserves this property, and because the sum of two such functions is another. In contrast, the subset of polynomials whose graph goes through the point $(1, 1)$ is not a subspace because it fails to satisfy all three of these same properties.

- **Extreme Subspaces:** Suppose $V$ is a vector space. By definition, if $S \subset V$ is a subset of $V$ that is closed under addition and scalar multiplication, we call that a subspace. However, students often fail to notice that this definition always includes a very small and very large subspace. For example, the smallest subspace of $V$ is the subset containing just the zero vector. And, the largest subspace of $V$ is $V$ itself. (Even though we call it a subspace, that does not rule out the possibility of it being the whole thing. Strange as it may sound to others, mathematicians consider $V$ to be a subset of itself.)
Null Spaces, Column Spaces, Kernels and Ranges

• Let \( A \) be an \( m \times n \) matrix. Associated to this matrix are two important subspaces! We will look at them first and then look at some analogous examples in the case of a different vector space.

• A Subspace of \( \mathbb{R}^m \) associated to \( A \): The matrix \( A \) has \( n \) columns \( A = [a_1 \ a_2 \ \cdots \ a_n] \) each of which is in \( \mathbb{R}^m \). So, one vector space associated to \( A \) is the column space or range which is \( \text{Col} A = \text{Span}\{a_1, \ldots, a_n\} \). Thinking of \( A \) as a transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) we can see why it is called the range. (To see a proof that this is indeed a subspace of \( \mathbb{R}^m \), take a look at homework exercise 29 which “walks you through it”.)

• A Subspace of \( \mathbb{R}^n \) associated to \( A \): We can show that the solution set of \( A x = 0 \) is a subspace of \( \mathbb{R}^n \). This is subspace is generally called the kernel or null space \( \text{Nul} A \). Here is how we would check that this works:

**Claim:** The solution set \( S \) of \( A x = 0 \) is a subspace of \( \mathbb{R}^n \).

**Proof:**  We need to check three things:
1) The zero vector \( 0 \) is in \( S \) because \( A 0 = 0 \).
2) Suppose \( v_1 \) and \( v_2 \) are any two elements of \( S \) and let \( w = v_1 + v_2 \) be their sum. Then
   \[
   Aw = A(v_1 + v_2) \quad \text{(the definition of } w) \\
   = Av_1 + Av_2 \quad \text{(distribute matrix multiplication)} \\
   = 0 + 0 \quad \text{(because } v_1 \text{ and } v_2 \text{ are in } S) \\
   = 0.
   \]
   So, \( S \) is closed under addition.
3) Suppose \( v \) is an element of \( S \) and \( \lambda \) is a scalar. Let \( w = \lambda v \). Then, since scalar multiplication commutes with matrix multiplication \( Aw = A(\lambda v) = \lambda Av = \lambda 0 = 0 \). This shows that \( S \) is closed under scalar multiplication.

• Note that this is not true for the non-homogenous equation \( A x = b \)! In particular, if \( b \neq 0 \) then the solution set to that equation does not include the zero vector and is not closed under addition or scalar multiplication!

• BTW: You don’t have to do any work to write \( \text{Col} A \) as a span (it is just the span of the columns of \( A \)), but it takes work to write \( \text{Nul} A \) that way (you have to actually solve the homogeneous equation).

• On the other hand, it is easy to check if a given vector is in \( \text{Nul} A \) (just multiply and see if you get zero), but harder to see if it is in \( \text{Col} A \) (you have to row reduce to see whether the augmented matrix has a pivot in the last column)!

• Now, let’s do the same thing for abstract vector spaces. First we need something like the matrix \( A \)!
**Definition:** If we have two vector spaces $V$ and $W$, then a linear transformation $T : V \rightarrow W$ is a function satisfying the linearity condition

$$T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2).$$

- Now what are the analogues of the column and null spaces?

**Definition:** The kernel (or null space) of a linear transformation is the subspace of all vectors that get mapped to the zero vector. The range (or column space) of a linear transformation is the subspace of all vectors in $W$ which appear as $T(v)$ for some $v$ in $V$.

NOTE: As this definition says, both of these are subspaces. (Why?)

**Question 1:** Consider the linear transformation $T : P_2 \rightarrow \mathbb{R}^2$ defined by

$$T(p(t)) = \begin{bmatrix} p(0) + p'(0) \\ p(0) - p'(0) \end{bmatrix}.$$

Then, for instance, $T(t^2 + 2t - 5) = [-3 - 7]^\top$. This automatically shows us that $[-3 - 7]^\top$ is in the range...but what else is in the range? Also, note that $T(t^2) = [0 0]^\top$, so $t^2$ is in the kernel...but what else is in the kernel? (See problems 31–32 for more examples like this.)

**Question 2:** What is the set of solutions to the equation $f''(x) + \omega f(x) = 0$? Note that it is the same as the kernel of the linear transformation $T(f) = f'' + \omega f$ on the vector space of functions. That means that it is a vector space. (In fact, you checked this in 4.1 problem 19!) Note also that when you do this in a differential equations class you take a determinant. Why? (Answer: to see if your solutions are independent!)

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**Homework**

**Section 4.2:** Read the section and do these problems. 1–6, 17–23, 24, 31, 32